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# On well $p$-embedded subgroups of finite groups

**Nataliya V. Hutsko and Alexander N. Skiba**

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Abstract. Let $G$ be a finite group, $H$ a subgroup of $G$ and $H_{sG}$ the subgroup of $H$ genarated by all those subgroups of $H$ which are $s$-permutable in $G$. Then we say that $H$ is well $p$-embedded in $G$ if $G$ has a quasinormal subgroup $T$ such that $HT = G$ and $T \cap H \leq H_{sG}$. In the present article we use the well $p$-embedded groups to obtain new characterizations for some class of finite soluble, supersoluble, metanilpotent and dispersive groups.

## Introduction

All groups under study in this article are finite. Ore considered [10] two generalizations of normality that still pique the unwaning interest of researchers. Note first of all that quasinormal subgroups were introduced in [10] into the practice of mathematicians for the first time. Following [10], we say that a subgroup $H$ of a groups $G$ is *quasinormal in* $G$ if $H$ commutes with every subgroup of $G$ (i.e. $HT = TH$ for all subgroups $T$ of $G$). It turned out that quasinormal subgroups possess a series of interesting properties [2, 6, 9, 10, 11, 16, 17] and that actually they are not much different from normal subgroups. Note, in particular, that according to [9] for each quasinormal subgroups $H$ we have $H^G/H_G \subseteq Z_\infty(G/H_G)$, and by [12, Theorem 2.1.3], quasinormal subgroups are precisely those subnormal subgroups of $G$ that are modular elements in the lattice of all subgroups of $G$.

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It is clear that if a subgroup $H$ of $G$ is normal in $G$, then $G$ must have some subgroup $T$ that satisfies the condition

$$G = HT \text{ and both subgroups } T \text{ and } T \cap H \text{ are normal in } G. \qquad (*)$$

Therefore, $(*)$ is another generalization of normality. This idea appeared firstly in [10] too, where it is shown in particular that *$G$ is soluble if and only if all maximal subgroups of $G$ satisfy* $(*)$ (in this regard, also see the article of Baer [1]). Later the subgroups satisfying $(*)$ were called *c-normal* in [18]. In this article a nice theory of $c$-normal subgroups was presented and some of its applications were given to the questions of classification of groups with some distinguished systems of subgroups.

Recall that a subgroup $H$ of $G$ is said to be $s$-permutable or $s$-quasinormal [10] in $G$ if $HP = PH$ for all Sylow subgroups $P$ of $G$.

In the present article we exemine the following concept which generalizes the conditions of quasinormality as well as $c$-normality for subgroups.

**Definiton 1.** Let $H$ be a subgroup of $G$. Then we say that $H$ is well $p$-embedded in $G$ if $G$ has a quasinormal subgroup $T$ such that $HT = G$ and $T \cap H \leq H_{sG}$.

In this definition $H_{sG}$ denotes the $s$-core of $H$ [14], that is the subgroup of $H$ genarated by all those subgroups of $H$ which are $s$-permutable in $G$.

It is clear that every $s$-permutable subgroup and $c$-normal subgroup are well $p$-embedded. The following simple example shows that, in general, a well $p$-embedded subgroup need not be quasinormal or $c$-normal.

**Example 1.** Consider $P = M_m(2) =< x, y | x^{2^{m-1}} = y^2 = 1, x^y = x^{1+2^{m-2}} >$, where $m > 3$, and take $A =< x >$ and $B =< y >$. Then $P = [A]B$ and $|B| = 2$. Since $Z(P)$ is a cyclic group of order $2^{m-2}$, it follows that $B$ is normal in $Z(P)B$. Given a group $Z_3$ of prime order 3, take $G = Z_3 \wr P = [K]P$, where $K$ is the base of the regular wreath product $G$. Since $G = (KB)A$, so $A \cap KB = 1$ and $P$ is a modular group. It follows that $KB$ is quasinormal in $G$. Hence $A$ is well $p$-embedded in $G$, but not quasinormal and not $c$-normal in $G$.

In the present article we use the well $p$-embedded groups to obtain new characterizations for some class of finite soluble, supersoluble, metanilpotent and dispersive groups.

## 1. Preliminaries

Let $G$ be a group and $p_1 > p_2 > \ldots > p_t$ are different prime divisors of the order of $G$. Then the group $G$ is said to be *dispersive* (in sence Ore [10]) if there are subgroups $P_1, P_2, \ldots, P_t$ such that $P_k$ is a Sylow $p_k$-subgroup of $G$ and the subgroup $P_1P_2 \ldots P_k$ is normal in $G$ for all $k = 1, 2, \ldots, t$.

The following known results about subnormal subgroups will be used in the paper several times.

**Lemma 1.1.** *Let $G$ be a group and $A \leq K \leq G$, $B \leq G$. Then*

*(1) If $A$ and $B$ are subnormal in $G$, then $\langle A, B\rangle$ is subnormal in $G$ [3, A, Lemma 14.4].*

*(2) Suppose that $A$ is normal in $G$. Then $K/A$ is subnormal in $G/A$ if and only if $K$ is subnormal in $G$ [3, A, Lemma 14.1].*

*(3) If $A$ is subnormal in $G$, then $A \cap B$ is subnormal in $B$ [3, A, Lemma 14.1].*

*(4) If $A$ is a subnormal Hall subgroup of $G$, then $A$ is normal in $G$ [19].*

*(5) If $A$ is subnormal in $G$ and $B$ is a Hall $\pi$-subgroup of $G$, then $A \cap B$ is a Hall $\pi$-subgroup of $A$ [19].*

*(6) If $A$ is subnormal in $G$ and $A$ is a $\pi$-subgroup of $G$, then $A \leq O_\pi(G)$ [19].*

*(7) If $A$ is subnormal in $G$ and $B$ is a minimal normal subgroup of $G$, then $B \leq N_G(A)$ [3, A, Lemma 14.5].*

*(8) If $A$ is a subnormal soluble (nilpotent) subgroup of $G$, then $A$ is contained in some soluble (respectively in some nilpotent) normal subgroup of $G$ [19].*

We will need to know a few facts about $s$-permutable subgroups.

**Lemma 1.2.** *[8] Let $G$ be a group and $H \leq K \leq G$. Then*

*(1) If $H$ is $s$-permutable in $G$, then $H$ is $s$-permutable in $K$.*

*(2) Suppose that $H$ is normal in $G$. Then $K/H$ is $s$-permutable in $G$ if and only if $K$ is $s$-permutable in $G$.*

*(3) If $H$ is $s$-permutable in $G$, then $H$ is subnormal in $G$.*

From Lemma 1.2 we directly have.

**Lemma 1.3.** *Let $G$ be a group and $H \leq K \leq G$. Then the following statements hold:*

*(1) $H_{sG}$ is a $s$-permutable subgroup of $G$ and $H_G \leq H_{sG}$.*

*(2) $H_{sG} \leq H_{sK}$.*

*(3) Suppose that $H$ is normal in $G$. Then $(K/H)_{s(G/H)} = K_{sG}/H$.*

*(4) If $H$ is either a Sylow subgroup of $G$ or a maximal subgroup of $G$, then $H_{sG} = H_G$.*

*Proof.* Statements (1-3) are evident. By Lemmas 2(1) and 3(1), $H_{sG}$ is subnormal in $G$ and so in the case when $H$ is a Sylow subgroup of $G$, $H_{sG} = H_G$, by Lemma 1(6).

Now assume that $H$ is a maximal subgroup of $G$. If $D = H_G \neq 1$, then by induction $(H/D)_{\pi(G/D)} = (H/D)_{(G/D)} = D/D$. Hence $H_{sG} = D$. Let $D = 1$ and let $N$ be a minimal normal subgroup of $G$. Then by [3], we know that either $N$ is the only minimal normal subgroup of $G$ and $C = C_G(N) \leq N$ or $G$ has precisely two minimal normal subgroups $N$ and $R$ say, $N \simeq R$ is non-abelian, $R = C$ and $N \cap H = 1 = R \cap H$. Let $L$ be a minimal subnormal subgroup of $G$ contained in $H$. If $L \leq N$, then $L^G = L^{NH} = L^H \leq D = 1$, a contradiction. Hence $L \nsubseteq N$ and analogously $L \nsubseteq R$. Hence $L \cap N = 1 = L \cap R$. But by Lemma 1(7), $NL = N \times L$, so $L \leq C$, a contradiction. Thus $H_{sG} = 1 = D$. □

**Lemma 1.4.** *Let $G$ be a group and $H \leq K \leq G$. Then*

*(1) Suppose that $H$ is normal in $G$. Then $K/H$ is well $p$-embedded in $G/H$ if and only if $K$ is well $p$-embedded in $G$.*

*(2) If $H$ is well $p$-embedded in $G$, then $H$ is well $p$-embedded in $K$.*

*(3) Suppose that $H$ is normal in $G$. Then the subgroup $HE/H$ is well $p$-embedded in $G/H$ for every well $p$-embedded in $G$ subgroup $E$ satisfying $(|H|, |E|) = 1$ .*

*Proof.* (1) *Necessity.* Suppose first that $K/H$ is well $p$-embedded in $G/H$ and let $T/H$ be a quasinormal subgroup of $G/H$ such that
$(K/H)(T/H) = G/H$ and $(T/H) \cap (K/H) \leq (K/H)_{s(G/H)}$. By Lemma 2(3), $T/H$ is subnormal in $G/H$. By Lemma 1(2), $T$ is subnormal in $G$ . Besides, we have $KT = G$ and $T \cap K \leq K_{sG}$, by Lemma 3(3). Hence $K$ is well $p$-embedded in $G$.

*Sufficiency.* Now assume that for some quasinormal subgroup $T$ of $G$ we have $KT = G$ and $T \cap K \leq K_{sG}$. Then by Lemma 1(1), $HT$ is subnormal in $G$, so by Lemma 1(2), $HT/H$ is subnormal in $G/H$. Besides, we have $(HT/H)(K/H) = G/H$ and $(HT/H) \cap (K/H) = (HT \cap K)/H = H(T \cap K)/H \leq HK_{sG}/H = K_{sG}/H = (K/H)_{s(G/H)}$, by Lemma 3(3). Thus $K/H$ is well $p$-embedded in $G/H$.

(2) Let $T$ be a quasinormal subgroup of $G$ such that $HT = G$ and $T \cap H \leq H_{sG}$. Then $K = K \cap HT = H(K \cap T)$ and $K \cap T$ is quasinormal in $K$. By Lemma 3(2), we also see that $(K \cap T) \cap H \leq H_{sG} \leq H_{sK}$. Hence $H$ is well $p$-embedded in $K$.

(3) Assume that $E$ is well $p$-embedded in $G$ and let $T$ be a quasinormal subgroup of $G$ such that $ET = G$ and $T \cap E \leq E_{sG}$. Clearly, $H \leq T$,

so $T \cap HE = H(T \cap E) \leq H(E_{sG}) \leq (HE)_{sG}$. Hence $HE$ is well $p$-embedded in $G$. By (2), $HE/H$ is well $p$-embedded in $G/H$. □

The following Lemmas will be necessary for the proof of theorems in Section 2.

**Lemma 1.5.** *If every maximal subgroup of group $G$ has complement, which is a quasinormal subgroup in $G$, then $G$ is nilpotent.*

*Proof.* Suppose that this is false and that $G$ is a counterexample of minimal order. Then $|G|$ is not prime, so $G$ is not simple group. Let $N$ be any proper normal subgroup of $G$ and $M/N$ a maximal subgroup in $G/N$. And let $T$ be a permutable subgroup in $G$ such that $G = MT$ and $M \cap T = 1$. Then $TN/N$ is permutable in $G/N$, $(TN/N)(M/N) = G/N$ and $(TN/N) \cap (M/N) = (TN \cap M)/N = N(T \cap M)/N = N/N$. As the class of all nilpotent groups is the saturated formation, we see that $G$ has only minimal normal subgroup. Let $N$ be only minimal normal subgroup of $G$. Then $C_G(N) = N$. Let $M$ be a maximal subgroup of group $G$ such that $N \leq M$. And let $T$ be permutable in $G$ such that $G = TM$ and $T \cap M = 1$. By Lemma 1(7), $N \leq N_G(T)$ and $NT = N \times T$. Then $T \leq C_G(N) = N$. The received contradiction finishes the proof of lemma. □

**Lemma 1.6.** *Suppose that $G = AB$ and $A$ is a subnormal subgroup of $G$, $B$ a nilpotent subgroup. If every Sylow subgroup of $A$ has a quasinormal complement in $G$, then $G$ is nilpotent.*

*Proof.* Suppose that this is false and let $G$ be a counterexample of minimal order. Then

(1) *$A$ and every proper subgroup of $G$ containing $A$ are nilpotent.*

Let $A \leq M \leq G$ with $M \neq G$. Then $M = M \cap AB = A(M \cap B)$, where $M \cap B$ is nilpotent in $G$, $A$ is a subnormal subgroup in $M$. Let $A_p$ be a Sylow subgroup of $A$ and $T$ a subnormal complement for $A_p$ in $G$. In view of Lemma 1(3), $M \cap T$ is subnormal in $M$, so $M = M \cap A_pT = A_p(M \cap T)$. Thus the hypothesis of the theorem is true for $M$. But $|M| < |G|$, contrary to the choice of $G$. Thus $M$ is nilpotent. Clearly, $A$ is nilpotent.

(2) *$G$ is soluble.*

By the condition, $A$ is subnormal in $G$. Then in view of (1) and Lemma 1(8), $A$ contains in some soluble normal subgroup $N$ of $G$. But $G/N \simeq B/B \cap N$ is nilpotent, so $G$ is soluble.

(3) *$G/P$ is nilpotent for every normal $p$-subgroup $P$ of $G$, containing Sylow $p$-subgroup of $A$.*

We shall show that the hypothesis of the theorem is true for $G/P$. Clearly, that $(AP/P)(BP/P) = G/P$, where $BP/P$ is nilpotent and

$AP/P$ a subnormal in $G/P$. Let $Q/P$ be a Sylow $q$-subgroup of $AP/P \simeq A/A \cap P$. Then $(q, |P|) = 1$ and $Q = A_qP$ for some Sylow $q$-subgroups $A_q$ of $A$. In view of (1), $A$ is nilpotent, so $A_q$ is subnormal in $G$ and $Q = A_q \times P$. Let $T$ be a subnormal complement for $A_q$ in $G$. Let $D = Q \cap TP = Q_1 \times P_1$, where $Q_1$ is a Sylow $q$-subgroup of $D$ and $P_1 \leq P$. Clearly, $Q_1 \leq A_q$. Since $(q, |P|) = 1$, $Q_1 \leq T_q$ for any Sylow $q$-subgroups $T_q$ of $T$ and therefore $Q_1 \leq T \cap A_q = 1$. Thus $D = P_1$ and hence $TP/P \cap Q/P = 1$. It follows that $TP/P$ is the subnormal complement for $Q/P$ in $G/P$. At the choice of $G$ we conclude that $G/P$ is nilpotent.

(4) *$A \leq F(G)$ and $F(G)$ is a $r$-group for some prime $r$.*

Let $P$ be a Sylow $r$-subgroup of $A$. Then in view of (1), $P$ is subnormal in $G$. By Lemma 1(6), $P \leq O_r(G)$. According to (3), $G/O_r(G)$ is nilpotent. Since $G$ is not nilpotent group, $A \leq F(G) = O_r(G)$.

(5) *$|G| = p^aq$ for some primes $p$ and $q$ and Sylow $p$-subgroup of $G$ is normal.*

Let $M$ be a normal subgroup of group $G$ such that $A \leq M$ and $G/M$ a simple group. In view of (2), $|G : M| = q$ is a prime. According to (1), $M$ is nilpotent. As every Sylow subgroup $P$ of $M$ is characteristic in $M$, $P$ is normal in $G$ and in view of (4), $M = P$.

(6) *$A$ is a $p$-group.*

It directly follows from (4) and (5).

*Final contradiction.*

Let $T$ be a subnormal complement to a subgroup $A$ in $G$. Then by Lemma 1(5), the Sylow $q$-subgroup $Q$ of $B$ contains in $T$. Let $D = AQ$. Then by Lemma 1(3), $T \cap D = Q(T \cap A) = Q$ is subnormal in $D$. Thus $D = A \times Q$, so $A \leq N_G(Q)$. Hence $B \leq N_G(Q)$. Then $Q$ is normal in $G$. Hence in view of (5), $G$ is nilpotent. The received contradiction finishes the proof of the lemma. □

**Lemma 1.7.** *If $G = AB$, where every Sylow subgroup of $A$ is well $p$-embedded in $G$ and $B$ is a Hall nilpotent subgroup in $G$, then $G$ is soluble.*

*Proof.* Suppose that this is not true and that $G$ is a counterexample of minimal order. Then every minimal normal subgroup of $G$ contained in $A$ is not abelian. Indeed, if for some abelian the minimal normal subgroup $L$ we have $L \leq A$, then by Lemma 4, the hypothesis of lemma is true for $G/L$. Consequently to the choice of group $G$, $G/L$ is metanilpotent. It then follows that $G$ is soluble, contrary to the choice of $G$.

Now assume that $A = G$ and let $P$ be any Sylow subgroup in $G$. Let $D = P_{qG}$. By Lemma 2(3), the subgroup $D$ is subnormal in $G$. By [13, II, Corollary 7.7.2], $D \leq F(G)$. But $G$ has not the abelian minimal normal subgroups and therefore $D = F(G) = 1$. According to the condition, a subgroup $P$ is well $p$-embedded in $G$, so $G$ has such permutable subgroup $T$ that is the complement to $P$ in $G$. It is clear that $T$ is subnormal in $G$ and consequently $T$ is a normal subgroup in $G$. Thus every Sylow subgroup of $G$ has normal complement in $G$. But then $G$ is a nilpotent group, a contradiction. □

**Lemma 1.8.** *Suppose that $G = [P]M$ and $P$ is a Sylow $p$-subgroup in $G$, $M$ is a soluble group. If all maximal subgroups of $P$ are well $p$-embedded in $G$, then $G$ is $p$-supersoluble.*

*Proof.* Suppose that this is not true and that $G$ is a counterexample of minimal order.

(1) *If $N$ is a minimal normal subgroup of $G$, then $G/N$ is a $p$-supersoluble group.*

Indeed, $G/N = [PN/N](MN/N)$, where $PN/N$ is a Sylow $p$-subgroup in $G/N$, $MN/N$ is a soluble group. Let $K/N$ be any maximal subgroup of $PN/N$.

We shall show that a subgroup $K/N$ is well $p$-embedded in $G/N$. Since $P$ is a Sylow $p$-subgroup in $G$, so $K = K \cap PN = N(K \cap P)$. We shall show first that $K \cap P$ is a maximal subgroups of $P$. Note that $K \cap P \neq P$. Indeed, if $K \cap P = P$, then $P \subseteq K$ and $K/N = PN/N$, contrary to the choice of $K/N$. Now assume that exists a subgroup $T$ such that $K \cap P \subset T \subset P$. Then $K = N(K \cap P) \subseteq TN \subseteq PN$. But $K$ is a maximal subgroup of $P$, so either $K = TN$ or $TN = NP$. If $K = TN$, then $T \subseteq K \cap P \subset T$ that is impossible. Hence $TN = NP$, so $P = P \cap TN = T(P \cap N) \subseteq T(P \cap K) = T$. This gives a contradiction. So $K \cap P$ is a maximal subgroup of $P$.

By condition of lemma, $K \cap P_p$ is well $p$-embedded in $G$. Thus by Lemma 4(2), $(K \cap P_p)N/N$ is well $p$-embedded in $GN/N$, so $K/N$ is a well $p$-embedded subgroup. Thus the hypothesis is still true for $G/N$. By the choice of $G$, $G/N$ is a $p$-supersoluble group.

(2) *$N$ is the only minimal normal subgroup of $G$ and $N$ is a $p$-group.*

Since the class of all $p$-supersoluble groups is the saturated formation (see [13, p. 35]), so $N$ is the only minimal normal subgroup of $G$. Since $G$ is $p$-supersoluble, so either $N$ is a $p'$-group or $N$ a $p$-group. If $N$ is a $p'$-group, then $G$ is $p$-supersoluble. Hence $N$ is a $p$-group.

(3) $N = P$.

Since $N \not\leq \Phi(G)$, there exists a subgroup $L$ of $G$ such that $G = [N]L$. We show that $N = O_p(G)$. Indeed, $O_p(G) = O_p(G) \cap NL = N(O_p(G) \cap L)$. Since $O_p(G) \leq F(G) \leq C_G(N)$, so $O_p(G) \cap L$ is normal in $G$. It follows that $O_p(G) \cap L = 1$. Hence $N = O_p(G) = P$.

*Final contradiction.*

Let $K$ be a maximal subgroup of $P$. Then by hypothesis, $G$ has a quasinormal subgroup $T$ such that $KT = G$ and $T \cap K \leq K_{sG}$. Since $K \leq N$, so $NT = G$. If $N \cap T = 1$, then $KT \neq G$. Hence $N \cap T \leq N$. If $N \cap T < N$, then we have a contradiction to the minimality of N. Thus $N \cap T = N$, so $N \leq T$ and $T = G$. But $K$ is well $p$-embedded in $G$, so $K \cap T = K \leq K_{sG}$. Hence $K$ is $s$-permutable in $G$, a contradiction. □

## 2. Characterizations of finite soluble, supersoluble, metanilpotent and dispersive groups

**Theorem 2.1.** *$G$ is soluble if and only if $G = AB$, where $A$, $B$ are subgroups of $G$ sutisfying every maximal subgroup of $A$ and every maximal subgroup of $B$ are well $p$-embedded in $G$.*

*Proof. Necessity.* Suppose that this is false and let $G$ be a counterexample of minimal order.

(1) *If $N$ is a minimal normal subgroup of $G$ contained in $A \cap B$, then $G/N$ is soluble* (it directly follows from Lemma 4(1)).

(2) $A \neq G \neq B$.

Indeed, let $A = G$. Let $R$ be a minimal normal subgroup of $G$. Then the hypothesis of our theorem is true for $G/R = (G/R)(G/R)$. In view of (1), $G/R$ is soluble. Thus $R$ is the only minimal normal subgroup of $G$, $R \nleq \Phi(G)$ and $R = A_1 \times \ldots \times A_t$, where $A_1 \simeq \ldots \simeq A_t$ is a simple non-abelian group. Let $p$ be a prime divisor of the order $|R|$ and $M$ a maximal subgroup of $G$ containing $N = N_G(P)$, where $P$ is a Sylow $p$-subgroup of $R$. Then by Frattini's Lemma, $G = RM$, so $M_G = 1$. Let $T$ be a quasinormal subgroup in $G$ such that $G = TM$ and $M \cap T \leq M_{sG}$. By Lemma 3(4), $M \cap T \leq M_{sG} = M_G = 1$. Hence $T$ is a complement for $M$ in $G$. Clearly, $p$ does not divide $|G : M|$, so $(p, |T|) = 1$. It follows that $T \cap R = 1$. By [3, A, Lemma 14.3], $TR = T \times R$. Since $R$ is the only minimal normal subgroup of $G$ and $R$ is not abelian, $T \leq C_G(R) = 1$. Hence $G = TM = M$. This is a contradiction.

(3) *$A$, $B$ are solube* (it follows from (2) and a choice of group $G$).

*Final contradiction.*

Let $R$ be a largest normal soluble subgroup of $G$. We shall show, that $AR/R$ is nilpotent. If $A \leq R$ it is obvious. Let now $A \nsubseteq R$ and $R \cap A \leq M$, where $M$ is the maximal subgroup of $A$. Let $T$ be a quasinormal subgroup of $G$ such that $G = MT$ and $M \cap T \leq M_{sG}$. Then $A = A \cap MT = M(A \cap T)$ and $A \cap T$ is a quasinormal subgroup in $A$. Since $T \cap M$ is a $s$-permutable subgroup in $G$, so by lemma 2(3), $T \cap M$ is a subnormal subgroup in $G$. In view of (3), $T \cap M$ is soluble. Hence

$T \cap M \leq R$. Then we have

$$(R \cap A)(T \cap A) \cap M = (R \cap A)(T \cap A \cap M) = (R \cap A)(T \cap M) \leq R \cap A.$$

Hence by Lemma 5, $A/R \cap A$ is nilpotent, so $AR/R \simeq A/R \cap A$ is nilpotent. It is similarly possible to show that $BR/R$ is nilpotent. Hence by [7, Theorem 3], $G/R = (AR/R)(BR/R)$ is soluble. Thus $G$ is soluble, a contradiction.

*Sufficiency.* Suppose $G$ is soluble and let $M$ be a maximal subgroup of group $G$. Then by [3, A, Theorem 15.6], $M/M_G$ has a normal complement in $G/M_G$ and therefore $M/M_G$ is well $p$-embedded in $G/M_G$. Thus by Lemma 4(1), $M$ is well $p$-embedded in $G$. □

**Corollary 1.** *$G$ is soluble if and only if all maximal subgroups are well $p$-embedded in $G$.*

**Theorem 2.2.** *$G$ is metanilpotent if and only if $G = AB$, where $A$ is a subnormal subgroup in $G$, $B$ is a Hall abelian subgroup in $G$ and every Sylow subgroup of $A$ is well $p$-embedded in $G$.*

*Proof. Necessity.* Suppose that this is false and let $G$ be a counterexample of minimal order. By Lemma 7, $G$ is soluble. Then following statements hold.

(1) *Let $N$ be a minimal normal subgroup in $G$, being $p$-subgroup for some prime $p$. If either $N \leq A$ or $(p, |A|) = 1$, then a quatient $G/N$ is metanilpotent.*

Clear, $A/N$ is subnormal in $G/N$, $BN/N \simeq B/B \cap N$ is a Hall abelian subgroup in $G/N$ and $G/N = (A/N)(BN/N)$. Let $P/N$ be a Sylow $q$-subgroup in $AN/N$. Let $Q$ be a Sylow subgroup in $AN$ such that $P = QN$. By [13, III, Lemma 11.6], $Q = A_qN_q$ for some Sylow $q$-subgroups $A_q$ of $A$ and for Sylow $q$-subgroups $N_q$ of $N$. Since group $G$ is soluble, $N$ is the abelian $p$-group for some prime $p$. And if either $N \leq A$ or $(p, |A|) = 1$, $A_qN/N$ is a Sylow $q$-subgroup in $AN/N$. By Lemma 4(1), $A_qN/N$ is well $p$-embedded in $G/N$. Thus the hypothesis of the theorem is true for $G/N$. Thus the quotient $G/N$ is metanilpotent according to the choice of $G$.

(2) *$P_{sG} = P_G$ for any Sylow $p$-subgroup $P$ of $A$* (it directly follows from Lemma 3(4)).

(3) $A_G \neq 1$.

Assume that $A_G = 1$. By hypothesis, $B$ is the abelian group, so $(A \cap B)^G = (A \cap B)BA = (A \cap B)^A \leq A$ and $A \cap B = 1$. Since $G = AB$ and by [13, III, Lemma 11.6], for any prime $p$ will be such Sylow $p$-subgroups $A_p$, $B_p$ and $G_p$ in $A$, $B$ and $G$, respectively, that $G_p = A_pB_p$.

Since $B$ is a Hall subgroup, it then follows from equality $A \cap B = 1$ that $A$ is a Hall subgroup in $G$. By hypothesis, $A$ is subnormal in $G$. In view of [13, II, Corollary 7.7.2 (1)], $A$ is normal in $G$. The received contradiction finishes the proof of the statement (3).

(4) *In $G$ there is the only minimal normal subgroup $L$ contained in $A$ and $L$ is a $p$-group for some prime number $p$.*

Indeed, by (3), one of the minimal normal subgroups $L$ of $G$ contains in $A$. Since the class of all metanilpotent groups is the saturated formation (see [13, II, p. 36]), $L$ is the only minimal normal subgroup of $G$ contained in $A$. But $G$ is soluble, so $L$ is a $p$-group for some prime $p$.

(5) *Every Sylow $q$-subgroup of $A$ has a quasinormal supplement in $G$ with $q \neq p$.*

Let $Q$ be a Sylow $q$-subgroup in $A$ with $q \neq p$. By hypothesis of our theorem, $G$ has a quasinormal subgroup $T$ such that $G = QT$ and $Q \cap T \leq Q_{sG}$. In view of (2) and (4), $Q_{sG} = 1$. Thus $T$ is a quasinormal supplement to $Q$ in $G$.

*Final contradiction.*

Let $A_p$ be a Sylow $p$-subgroup in $A$ and $P = (A_p)_{sG} = A_G$. We shall consider a quotient group $G/P = (A/P)(BP/P)$. By hypothesis, $G$ has a quasinormal subgroup $T$ such that $TA_p = G$ and $T \cap A_p \leq P$. Then $(A_p/P)(TP/P) = G/P$ and $A_p/P \cap TP/P = P(A_p \cap T)/P = P/P$, so $TP/P$ is a quasinormal supplement to $A_p/P$ in $G/P$. On the other hand, if $Q/N$ is a Sylow $q$-subgroup in $A/N$ with $q \neq p$, then in view of (5), $Q/P$ has a quasinormal supplement in $G/P$ (see the proof of the statement (3) Lemmas 6). Thus by Lemma 6, $G/P$ is nilpotent. Hence $G$ is metanilpotent. The received contradiction finishes the proof of the metanilpotently of $G$.

*Sufficiency.* Suppose that $G$ is metanilpotent. We shall show that every Sylow subgroup of $G$ is well $p$-embedded in $G$. Suppose that is false and let $G$ be a counterexample of minimal order. Then $G$ has a Sylow subgroup $P$ which is not well $p$-embedded in $G$. Let $N$ be any minimal normal subgroup in $G$ and $F$ is a Fitting subgroup of $G$. Suppose that $N \leq P$. Then $P/N$ is well $p$-embedded in $G/N$. By Lemma 4(1), $P$ is well $p$-embedded in $G$, a contradiction.

Thus $P_G = 1$, so $F \cap P \leq P_{sG} = P_G = 1$. Since $G$ is metanipotent and $FP/F$ is a Sylow subgroup in $G$, we see that $FP/F$ has a normal supplement $T/F$ in $G/F$. But $F$ and $T/F$ are $p'$-groups, so $T$ is a normal supplement to $P$ in $G$. Hence $P$ is well $p$-embedded in $G$. The received contradiction shows that every Sylow subgroup of $G$ is well $p$-embedded in $G$. □

**Corollary 2.** *$G$ is metanilpotent if and only if every Sylow subgroup is well $p$-embedded in $G$.*

**Theorem 2.3.** *Suppose that $G = AB$ and $A$ is a quasinormal subgroup in $G$, $B$ is a dispersive. If every maximal subgroup of any non-cyclic Sylow subgroup of $A$ is well $p$-embedded in $G$, then $G$ is dispersive.*

*Proof.* Suppose that this theorem is not true and let $G$ be a counterexample of minimal order.

(1) *Every proper subgroup $M$ of $G$ containing $A$ is dispersive.*

Let $A \leq M \leq G$ and $M \neq G$. Then $M = M \cap AB = A(M \cap B)$, where $M \cap B$ is dispersive and $A$ is $s$-quasinormal in $M$. By Lemma 4(2), any maximal subgroup of every non-cyclic Sylow subgroup of $A$ is well $p$-embedded in $M$ and $|M| < |G|$, then by the choice of group $G$, we have (1).

(2) *Let $H$ be not uniqueal normal subgroup in $G$ being $p$-group for some prime $p$. Suppose either $H$ contains a Sylow $p$-subgroup $P$ of $A$ or $P$ is cyclic or $H \leq A$. Then $G/H$ is dispersive.*

If $A \leq H$, then $G/H = BH/H \simeq B/B \cap H$ is dispersive. Let now $A \nsubseteq H$. Since $|G/H| < |G|$, we need to be shown that hypothesis of the theorem is true for $G/H$. Clearly, $G/H = (HA/H)(BH/H)$, where $HA/H$ is $s$-quasinormal in $G/H$ and $BH/H$ is dispersive. Let $Q/H$ be a Sylow $q$-subgroup of $AH/H$ and $M/H$ any maximal subgroup in $Q/H$. Let $Q_1$ be a Sylow $q$-subgroup of $Q$ such that $Q = HQ_1$. Clearly, $Q_1$ is a Sylow $q$-subgroup of $AH$. Thus $Q = A_qH$ for some Sylow $q$-subgroup $A_q$ of $A$. Assume that $Q/H$ is not a cyclic subgroup. Then $A_q$ is not cyclic. We shall show that $M/H$ is well $p$-embedded in $G/H$. If $H \leq A$, it directly follows from Lemma 4. Admit that either Sylow $p$-subgroup $P$ of $A$ cyclic or $P \leq H$. Then $p \neq q$. We shall show $M \cap A_q$ is maximal in $A_q$. Since $M \neq Q$ and $A_qH = Q$, we see that $M \cap A_q \neq A_q$. Assume that for some subgroup $T$ of $G$ we have $M \cap A_q \leq T \leq A_q$, where $M \cap A_q \neq T \neq A_q$. Then $M = H(M \cap A_q) \leq HT \leq HA_q = Q$. Since $M$ is maximal in $Q$, or $M = TH$ or $TH = HA_q$. If $M = TH$, then $T \leq M \cap A_q$, contrary to the choice of $T$. Thus $TH = HA_q$ and we have $A_q = A_q \cap TH = T(A_q \cap H) \leq T(M \cap A_q) = T$, a contradiction. Hence $M \cap A_q$ is a maximal subgroup in $A_q$. By hypothesis, $M \cap A_q$ is well $p$-embedded in $G$. Therefore $M/H = (M \cap A_q)H/H$ is well $p$-embedded in $G/H$. Hence the conditions of the theorem are true for $G/H$.

(3) *If $p$ is a prime and $(p, |A|) = 1$, then $O_p(G) = 1$.*

Let $H = O_p(G) \neq 1$. Then in view of (2), $G/H$ is dispersive. On the other hand, if $\pi$ is a set of all prime divisors $|A|$, then in view of [10] and [13, II, Corollary 7.7.2], $A \leq E$, where $E$ is a normal $\pi$-subgroup in $G$. Thus $G/E \simeq B/B \cap E$ is dispersive. But then $G \simeq G/H \cap E$ is dispersive, the contradiction.

(4) *$G$ is soluble.*

By hypothesis, $A$ is $s$-quasinormal in $G$. In view of [10] and [13, II, Corollary 7.7.2], $A$ contains in some soluble normal subgroup $E$ of $G$. Since $G/E \simeq B/B \cap E$ is dispersive, $G$ is soluble.

(5) $A_G \neq 1$.

Suppose that $A_G = 1$. Then by [8], $A$ is nilpotent. Let $P$ be a Sylow $p$-subgroup of $A$. Since $A$ is subnormal in $G$, so $P$ is subnormal in $G$. Thus by [13, II, Corollary 7.7.2], $P \leq O_p(G)$. But in view of (2), $G/O_p(G)$ is dispersive. By the choice of $G$, $P = A$. Let $q$ be a smallest prime divisor $|G/O_p(G)|$. Then $G$ has a normal maximal subgroup $M$ such that $P \leq M$ and $|G : M| = q$. Let $r$ be a largest prime divisor $|G|$ and $R$ be a Sylow $r$-subgroup of $M$. Then in view of (1), $R$ is normal in $M$, so $R \triangleleft G$. If $r \neq q$, $R$ is a Sylow $r$-subgroup of $G$ and $G/R$ dispersive. It follows that $G$ is dispersive, a contradiction. Hence $r = q$. But then $G/O_p(G)$ is a $r$-group. Let $B_r$ be a Sylow $r$-subgroup in $B$. Then $B_r$ is a Sylow $r$-subgroup in $G$. Since $AB_q$ is a subgroup of $G$ and in view of (1), we have $AB_q$ is dispersive and $B_q \triangleleft AB_q$. As $B$ is dispersive, $B_q \triangleleft B$ and $B_q \triangleleft G$. Hence $G$ is dispersive. The received contradiction proves (5).

*Final contradiction.*

Let $H$ be a minimal normal subgroup of $G$ containing in $A$. Let $H$ be a $p$-group and $P$ a Sylow $p$-subgroup of $A$. In view of (2), $G/H$ is dispersive. Let $q$ be a smallest prime divisor $|G/H|$. Then $G$ has a normal maximal subgroup $M$ such that $P \leq M$ and $|G : M| = q$. Let $r$ be a largest prime divisor $|G|$, $R$ be a Sylow $r$-subgroup of $M$. Then in view of (1), $R$ is normal in $M$ and so $R \triangleleft G$. As above we see $r = q$. Then $G/H$ is a $r$-group. Thus $H = A$. By Theorem 1.4 in [15], $G$ is dispersive, a contradiction. □

**Theorem 2.4.** *If $G = AB$, where $A$ is a subnormal subgroup in $G$ and $B$ is a Hall subgroup in $G$, which all Sylow subgroups are cyclic groups and any maximal subgroup of every non-cyclic Sylow subgroup of $A$ is well $p$-embedded in $G$, then $G$ is supersoluble.*

*Proof.* Suppose that this is false and that $G$ is a counterexample of minimal order.

(1) *Each proper subgroup $M$ of $G$ containing $A$ is supersoluble.*

Let $A \leq M \leq G$ and $M \neq G$. Then $M = M \cap AB = A(M \cap B)$, where $M \cap B$ is nilpotent and $A$ is a subnormal in $M$. By Lemma 4(2), any maximal subgroup of every non-cyclic Sylow subgroup of $A$ is well $p$-embedded in $M$ and $|M| < |G|$, then by the choice of group $G$, we have (1).

(2) *Let $H$ be a non-uniqueal normal subgroup in $G$. Suppose that $H$ is a $p$-group. Admit that $H$ contains Sylow $p$-subgroup $P$ of $A$ or $P$ is cyclic or $H \leq A$. Then $G/H$ is supersoluble* (see the proof of the statement (2) Theorems 2.3).

(3) *One of the Sylow subgroup of $A$ is not cyclic.*

Indeed, easily to see, that any Sylow subgroup of $G$ contains or in some subgroup interfaced with $A$ or in some subgroup interfaced with $B$. If all Sylow subgroups of $A$ are the cyclic groups, then every Sylow subgroup of $G$ is cyclic. But then by [5, VI, Theorem 10.3], $G$ is supersoluble, contrary to the choice of $G$.

(4) *$G$ is soluble.*

Assume that $A \neq G$. Then by view of (1), $A$ is supersoluble. By [13, II, Corollary 7.7.2 (4)], $A$ contains in some normal soluble subgroup $R$ of $G$. But $G/R = RB/R \simeq B/B \cap R$ is supersoluble group, so $G$ is soluble.

Now assume that $A = G$. If there is such prime $p$ and such maximal subgroup $M$ in some Sylow subgroup $G_p$ of $G$ that $M_{sG} \neq 1$, then $O_p(G) \neq 1$, this attracts resolvability of group $G$ in view of (2). Thus we can assume that for any Sylow subgroup $G_p$ of $G$ and for its any maximal subgroup $M$ we have $M_{sG} = 1$. Then $M$ has a quasinormal supplement $T$ in $G$ and the order Sylow $p$-subgroup of $T$ is equal $p$. By Lemma 4(2), condition of the theorem is true for $T$. Then by view of the choice of group $G$, $T$ is supersoluble. But it again attracts resolvability of group $G$.

(5) *$A$ is supersoluble.*

Let $A = G$ be a soluble group in which for any non-cyclic Sylow subgroup $G_p$ all its maximal subgroups are well $p$-embedded in $G$. Since the class of all supersoluble groups is the saturated formation (see [13, p. 35]), there is the only minimal normal subgroup $N$. Thus $N = C_G(N) \nsubseteq \Phi(G)$. By [5, III, Lemma 3.3(a)], $N \nsubseteq \Phi(G_p)$. Since $N \nsubseteq \Phi(G)$, so $G = [N]E$ for some maximal subgroup $E$ of $G$. Thus $M_{sG}E = EM_{sG}$. But $N \nsubseteq M$, so $M_{sG} \neq N$. If $M_{sG} \neq 1$, in view of maximality of a subgroup $E$, then $M_{sG} = G$, that attracts $N = N \cap M_{sG}E = M_{sG}(N \cap E) = M_{sG}$, a contradiction. Hence $M_{sG} = 1$ and $M$ has a quasinormal supplement $T$ in $G$.

It is clear that the order Sylow $p$-subgroup of $T$ is equal $p$. Hence in view of Lemma 4(2), the condition of the theorem is true for $T$. By the choice of group $G$, $T$ is a supersoluble group. Let $q$ be a largest prime divisor of the order of $T$. And let $T_q$ be a Sylow $q$-subgroup in $T$. We shall admit that $q \neq p$. Then $T_q$ is a Sylow $q$-subgroup in $G$. Since $T$ is subnormal in $G$, so $T_q \lhd G$. Then $T_q \leq C_G(N) = N$, a contradiction. Hence $q = p$ is the largest prime divisor of the order of $G$. In view of [13, I, Lemma 3.9], $O_p(G/C_G(N)) = O_p(G/N) = 1$. Hence by view of (2),

$N = G_p$, a contradiction.

(6) $A_G \neq 1$.

Let $p$ be a largest prime divisor of the order of $A$ and $A_p$ be a Sylow $p$-subgroup in $A$. By (5), a group $A$ is supersoluble and $A_p \lhd A$. By [13, II, Corollary 7.7.2 (1)], $A_p \leq O_p(G)$. In view of (2), $G/O_p(G)$ is a supersoluble group and $O_p(G)$ non-cyclic group by the choice of group $G$. It follows that $A_p \not\subseteq B^x$ for all $x \in G$. Therefore $A_p$ is a Sylow subgroup in $G$, so $A_p = O_p(G)$.

(7) *Let $N$ be a minimal normal subgroup of group $G$ contained in $A$. Then $N = A_p = G_p$ is a Sylow subgroup in $G$, where $p$ is the largest prime divisor of the order of $A$.*

Let $N$ be a minimal normal subgroup of $G$ contained in $A$. And let $p$ be the largest prime divisor of $A$. If $p$ divides $|B|$, $G_p \leq B$, where $G_p$ is a Sylow $p$-subgroup of $G$. By the condition, $G_p$ is a cyclic group. But $N \leq G_p$, so $N$ is a cyclic group. In view of (2), $G$ is supersoluble. The received contradiction with a choice of group $G$ shows, that $p$ does not divide $|B|$. Thus in view of (5), $O_p(G) = O_p(A) = A_p$, where $A_p$ is a Sylow $p$-subgroup of $A$. Since $O_p(A) \subseteq C_G(N) = N$, we have $N = A_p$ is a Sylow subgroup in $G$.

(8) *$G$ is $p$-supersoluble* (it directly follows from Lemma 8).

*Final contradiction.*

By (2), $G/N$ is supersoluble. By (8), $|N| = p$. Hence $G$ is supersoluble. The received contradiction finishes the proof of the theorem. □

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## Contact information

**N. V. Hutsko** Gomel State University of F.Skorina, Belarus, 246019, Gomel, Sovetskaya Str., 104
*E-Mail:* GutskoN@tut.by

**A. N. Skiba** Gomel State University of F.Skorina, Belarus, 246019, Gomel, Sovetskaya Str., 104
*E-Mail:* Skiba@gsu.unibel.by

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