Algebra and Discrete Mathematics RESEARCH ARTICLE
Number 2. **(2008).** pp. 42 – 49

Baer semisimple modules and Baer rings

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Communicated by A. V. Mikhalev

Abstract. We consider Baer rings and Baer semisimple R-modules which are generalizations of semisimple modules. Several characterization theorems of Baer semisimple modules are obtained. In particular, we prove that a ring R is a Baer ring if and only if R itself, regarded as a regular R-module, is Baer semisimple.

Throughout this paper, R is an associative ring with identity 1 and all R-modules are unital. Denote the set of idempotents of R by $E(R)$. Let M be a left R-module and a right S-module. Also, let X be a subset of M, R or S, respectively. Then we denote the left [resp. right] annihilator of X by $ann_\ell(X)$ [resp. $ann_r(X)$]. We also write $ann_\ell(\{m\})$ [resp. $ann_r(\{m\})$] by $ann_\ell(m)$ [resp. $ann_r(m)$].

We call a ring R a *Baer ring* if the left annihilator of any subset of R is generated by an idempotent. The properties of Baer rings and its generalizations have been studied by many authors, for example, see ([3], [4], [11] and [13]). We observe that Baer rings can be generalized into other forms, for example, rpp rings, etc. The rpp-rings and their generalizations have been extensively studied in the literature after Hattori (see, [2]-[15]). Recently, the authors have introduced the concept of right perpetual ideals and consequently, reduced pp rings are characterized by using right perpetual submodules (see [8]).

The research of the first author is jointly supported by the Natural Science Foundation of Jiangxi Province; the Science Foundation of the Education Department of Jiangxi Province and the Foundation of Jiangxi Normal University, China. The research of the second author is partially supported by a UGC (HK) grant (#2160297/05-06)

2000 Mathematics Subject Classification: *16W60.*

Key words and phrases: *Baer module; Baer semisimple module; perpetual submodule; Baer ring.*

Recall that a right ideal I of R is a *right perpetual ideal* of R if for every $x \in I$ and $y \in R$, $ann_\ell(x) \subseteq ann_\ell(y)$ implies that $y \in I$ (see [8]). Clearly, for any $X \subseteq R$, there exists the smallest right perpetual ideal of R containing X. We usually call this smallest right perpetual ideal of R containing X the *right perpetual ideal* generated by X and is denoted by $R^*(X)$. If $X = \{a\}$, then we write $R^*(X) = R^*(a)$.

The following results are known.

Lemma 1. [8] *The following statements hold in a ring R:*

(1) *If $e \in E(R)$, then $R^*(e) = eR$.*

(2) *For all $X \subseteq R$, $ann_r(X)$ is a right perpetual ideal of R.*

(3) *A ring R is lpp if and only if for any $a \in R$, $R^*(a)$ is generated by an idempotent.*

Let M be a right R-module. Denote the ring of R-endomorphisms of M by $End(M_R)$. If $End(M_R)$ is regarded as a set of left operations, in notation, $End_\ell(M_R)$, then M can be regarded as a left $End_\ell(M_R)$- right R-module. Inspiring by the definition of right perpetual ideals, we now define the perpetual submodules.

Definition 1. *Let M be a right R-module. Then, we call a (right R-)submodule N of M a perpetual submodule of M if for all $x \in N$ and $y \in M$, $ann_\ell(x) \subseteq ann_\ell(y)$ implies $y \in N$.*

It is clear that M and (0) are both trivial perpetual submodules of M. Also, the intersection of perpetual submodules of M is still a perpetual submodule of M and hence, there exists the smallest perpetual submodule of M containing X for $X \subseteq M$. Denote the smallest perpetual submodule of M containing X by $SM^*(X)$. On the other hand, if R is regarded as a regular right R-module R_R, then the left $End(R_R)$-right R-module R becomes a regular bimodule ${}_{End(R_R)}R_R$. Thus in this case, every perpetual submodule of R is a right perpetual ideal of R (same as in rings).

The following lemma can be easily proved.

Lemma 2. *Let M be a right R-module and $X \subseteq End_\ell(M_R)$. Then*

(1) *$ann_r(X)$ is a perpetual submodule of M.*

(2) *If $\varphi^2 = \varphi \in End_\ell(M_R)$, then φM is a perpetual submodule of M.*

The proof of the following lemma is straightforward.

Lemma 3. *Let M be a right R-module and $x \in M$. Then $SM^*(x) = ann_r(ann_\ell(x))$.*

The following result lemma is crucial in this paper but the proof can be found in [1].

Lemma 4. [1] *A R-submodule K of the right R-module M is a direct summand of M if and only if $K = eM$ for some idempotent $e \in End_\ell(M_R)$.*

Now, we formulate the following definition.

Definition 2. *Let M be a right R-module. Then*

(1) *M is called a **Baer simple R-module** if $M \neq 0$, and M contains no perpetual submodules of M other than M itself and (0).*

(2) *M is called a **Baer semismiple R-module** if every perpetual submodule of M is a direct summand of M.*

Evidently, a Baer simple R-module is itself Baer semisimple and the usual semisimple R-module is also Baer semisimple. Indeed, if M is a semisimple R-module, then every R-submodule N of M is a direct summand of M. By Lemma 4, $N = eM$, for some $e^2 = e \in End_\ell(M_R)$. This implies that every R-submodule of M is a perpetual submodule of M. Thus M is Baer semisimple.

Proposition 1. *Let M be a Baer semisimple R-module and N a perpetual submodule of M. Then the following statements hold:*

(i) $N = eM$ for some idempotent $e \in End_\ell(M_R)$.

(ii) N is Baer semisimple.

Proof. (i) By our hypothesis, M is Baer semisimple and hence, N is a direct summand of M. Now, by Lemma 4, $N = eM$, for some idempotent $e \in End_\ell(M_R)$.

(ii) It suffices to verify that any perpetual submodule of N is still a perpetual submodule of M. In other words, we only need to prove that the smallest perpetual submodule $SM^*_M(x)$ of M containing x is the smallest perpetual submodule $SM^*_N(x)$ of N containing x, for all $x \in N$. By Lemma 5, we have $N = eM$, for some idempotent $e \in End_\ell(M_R)$. Denote the left annihilator of K related to the R-module M and related to the R-module N by $ann^M_\ell(K)$ and $ann^N_\ell(K)$, respectively. Now, by Lemma 3, $SM^*_M(x) \subseteq N$. Let f be an idempotent endomorphism in $End_\ell(M_R)$ such that $SM^*_M(x) = fM$. Then, $fM \subseteq eM$. Thus, for any $x \in M$,we have

$$fx = ey = eey = efx \quad (y \in M),$$

and thereby, $f = ef$. Hence, fe is an idempotent endomorphism in $End_\ell(M_R)$ and also

$$fM = ffM \subseteq fefM \subseteq feM \subseteq fM,$$

that is, $fM = feM$. On the other hand, since the restriction $fe|_{eM}$ of $fe(= efe)$ to eM is an idempotent R-endomorphism which maps eM

into itself, we have $fM = feM = fe(eM)$ and hence, by Lemma 2, fM is a perpetual submodule of N. Now, by the minimality of $SM_N^*(x)$, we have $SM_N^*(x) \subseteq SM_M^*(x)$.

Now let $\varphi \in ann_\ell^N(x)$. Then, it can be easily observed that $M = eM \oplus (1-e)M$. Hence, we can define a mapping

$$\overline{\varphi} : M \to M; \quad y \mapsto \varphi(ey),$$

which is a R-homomorphism of M into itself with $\overline{\varphi}|_N = \varphi$. Clearly, $\overline{\varphi} \in ann_\ell^M(x)$. If $y \in SM_M^*(x)$, then by Lemma 3, $\psi(y) = 0$ for all $\psi \in ann_\ell^M(x)$, and furthermore, $\overline{\varphi}y = 0$, for all $\varphi \in ann_\ell^N(x)$. Note that $SM_M^*(x) \subseteq N$ and $\overline{\varphi}|_N = \varphi$. Thus $\overline{\varphi}y = 0$ implies that $\varphi y = 0$. This shows that $ann_\ell^N(x) \subseteq ann_\ell^N(y)$. Consequently, we can deduce $y \in SM_N^*(x)$, by Lemma 3. This leads to $SM_M^*(x) \subseteq SM_N^*(x)$. Thus, $SM_N^*(x) = SM_M^*(x)$, as required. □

The following is a characterization theorem for the Baer simple R-modules.

Theorem 1. *Let M be a Baer semisimple R-module and N a perpetual R-submodule of M. Then N is Baer simple R-module if and only if $N = eM$, for some primitive idempotent $e \in End_\ell(M_R)$.*

Proof. Suppose that N is a Baer simple R-submodule of M. Then, by Lemma 4, $N = eM$ for some idempotent $e \in End_\ell(M_R)$. Now let $f^2 = f \in End_\ell(M_R)$ such that $f \leq e$, i.e., $f = ef = fe$. Then $fM \subseteq eM$. Since N is Baer simple, $fM = (0)$ or $fM = eM$.

- If $fM = (0)$, then $f = 0$.
- If $fM = eM$, then for all $x \in M$,

  $$e(x) = f(y) = ff(y) = fe(x) = f(x) \quad (y \in M),$$

  and whence $e = f$.

This shows that e is a primitive idempotent of $End_\ell(M_R)$. Conversely, we assume that $N = eM$, where e is a primitive idempotent of $End_\ell(M_R)$. Then N is a perpetual submodule of M. Let K be a perpetual submodule of N. Now, by using the proof of Proposition 1, we can show that K is still a perpetual submodule of M, and by Lemma 4, $K = fM$ for some idempotent $f \in End_\ell(M_R)$. Now, $fM \subseteq eM$ implies that for all $x \in M$, we have

$$f(x) = e(y) = ee(y) = ef(x) \quad (y \in M),$$

and thereby, $f = ef$. By routine verification, fe is an idempotent of $End_\ell(M_R)$, and $fe \leq e$. But since e is primitive, $fe = e$ or $fe = 0$.

- If $fe = e$, then
$$fM \subseteq eM = feM \subseteq fM,$$
that is, $K = N$.

- If $fe = 0$, then
$$K = fM = f(fM) \subseteq f(eM) = (0).$$

This shows that the submodule N is indeed Baer simple. □

We next establish a "Schur Lemma " for Baer simple modules.

Theorem 2. *(Schur Lemma) If M is a Baer simple R-module, then $End_\ell(M_R)$ is a domain (such a ring satisfies the cancellative law).*

Proof. It suffices to show that any $\varphi \in End_\ell(M_R)\backslash\{0\}$ is injective. For this purpose, we only need to prove that $ann_r(\varphi) = (0)$. By Lemma 2, $ann_r(\varphi)$ is a perpetual submodule of M and, since M is Baer simple, $ann_r(\varphi) = M$ or $ann_r(\varphi) = (0)$. But since $\varphi \neq 0$, it is clear that $ann_r(\varphi) \neq M$. Thus $ann_r(\varphi) = (0)$ and hence φ is injective. □

Lemma 5. *Any nonzero Baer semisimple R-module M contains a Baer simple R-module.*

Proof. Without loss of generality, we may assume that M is not a Baer simple R-module. Then we can pick a nonzero element x of M such that $SM^*(x) \subset M$. By Lemma 4, $SM^*(x) = eM$ for some idempotent endomorphism $e \in End_\ell(M_R)$. By Lemma 1, $K = (1-e)M$ is a perpetual submodule of M not containing x. Now, by Zorn's lemma, there exists a perpetual submodule N of M which is maximal with respect to the property that $x \notin N$. Choose a perpetual submodule N' of M such that $M = N \oplus N'$ (by Lemma 4). Then, we can finish our proof by showing that N' is Baer simple. Indeed, if N'' is a nonzero perpetual submodule of N', then by Proposition 1, N' is Baer semisimple and $N' = N'' \oplus N'''$, where N''' is a submodule of N'. Thus $N \oplus N''$ is a direct summand of M. Again by Lemma 4, $N \oplus N'' = fM$ for some idempotent $f \in End_\ell(M_R)$ and by Lemma 2, $N \oplus N''$ is a perpetual submodule of M containing x (by the maximality of N) and $N \oplus N'' = M$, which implies that $N'' = N'$, as desired. □

Proposition 2. *A Baer semisimple module is the direct sum of a family of Baer simple submodules.*

Proof. Assume that M is a Baer semisimple module. Denote by A the set of Baer simple submodules of M. Then, we consider the subset $B \subset A$ with the following conditions:

- $\sum_{J \in B} J$ is a direct sum.
- $\sum_{J \in B} J$ is a perpetual submodule of M.

By Lemma 5, $A \neq \emptyset$. Now, by Zorn's lemma, we can consider the family of all the above B's with respect to the set inclusion. Thus we can pick a B to be the maximal element. For such a B, we can construct a perpetual submodule $M_1 := \oplus_{J \in B} J$. Now, by our hypothesis, $M = M_1 \oplus M_2$, where M_2 is a submodule of M. By Lemma 4, M_2 is a perpetual submodule of M and by Proposition 1, M_2 is a Baer semisimple, and hence by Lemma 5 again, $M_2 = K \oplus Q$, where K is a Baer simple submodule of M_2 and Q a submodule of M_2. Thus $M_1 \oplus K$ is a direct summand of M and of course, $M_1 \oplus K$ is a perpetual submodule of M, by Lemma 4. On the other hand, by using the proof of Proposition 1(ii), we can show that K is Baer simple in M. This contradicts the maximality of B. Therefore $M = M_1 = \oplus_{J \in B} J$. □

Theorem 3. *Let M be a R-module and P the set of submodules of the form eM, with $e \in E(End_\ell(M))$. Order the set P by set inclusion. Then the following statements are equivalent:*
(i) M is Baer semisimple.
(ii) The following two conditions hold:
(a) For any $x \in M$, $SM^*(x)$ is a direct summand of M.
(b) P forms a complete lattice.

Proof. $(i) \Rightarrow (ii)$ Since condition (a) holds trivially, we need only to show that condition (b) holds. Let $T \subseteq P$. Since every element of P is a perpetual submodule of M, $\bigcap_{J \in T} J$ is a perpetual submodule of M. By our hypothesis, $\bigcap_{J \in T} J$ is a direct summand of M. By Lemma 4, we have $\bigcap_{J \in T} J \in P$. Consider the smallest perpetual submodule K of M containing J with $J \in T$. It is clear that K is a direct summand of M, and whence $K \in P$. Thus K can be viewed as $sup_{J \in T}(J)$. Thus, P indeed forms a complete lattice.

$(ii) \Rightarrow (i)$ Assume that (ii) holds. Let I be a perpetual submodule of M. Consider $I = \bigcup_{x \in I} SM^*(x)$. Then by condition (a), $I = \bigcup_{x \in I} e_x M$, where e_x is the idempotent of $End_\ell(M)$ such that $SM^*(x) = e_x M$, for any $x \in I$. By condition (b), $I = eM$ for some $e \in E(End_\ell(M))$, that is, I is a direct summand of M. Consequently, M is a Baer semisimple module. □

Recall that [14, Lemma 2.3] a ring R is Baer if and only if R is lpp and under set inclusion, the set of all idempotent-generated principal right ideals forms a complete lattice. By using Lemma 1 and Theorem 3, we deduce the following characterization theorem of Baer rings.

Theorem 4. *A ring R is a Baer ring if and only if R itself, regarded as a regular R-module, is a Baer semisimple module.*

References

[1] F.W. Anderson and K.R. Fuller, *Rings and categories of modules (2nd edition)*, Graduate Texts in Mathematics, Springer-Verlag, Berlin, Heidelberg, New York, 2004.

[2] E.P. Armendariz, *A note on extensions of Baer and PP rings*, J. Austral. Math. Soc., Vol.18, 1974, pp.470-473.

[3] G.F. Birkenmeier, *Decompositions of Baer-like rings*, Acta Math. Hungary, Vol.59, N.3-4, 1992, pp.319-326.

[4] G.F. Birkenmeier, *Baer rings and quasicontinuous rings having a MDSN*, Pacific J. Math., Vol.97, N.2, 1981, pp.283-292.

[5] S. Endo, *Note on P.P. rings (a supplement to Hattori's paper)*, Nagoga Math. J., Vol.17, 1960, pp.167-170.

[6] J.A. Fraser and W.K. Nicholson, *Reduced pp rings*, Math. Japonica, Vol.34, 1989, pp.715-725.

[7] X.J. Guo and K.P. Shum, *l.p.p.-rings which are left *-semisimple*, International Mathematical Forum, Vol.2, N.56, 2007, pp.2787-2798.

[8] X.J. Guo and K.P. Shum, *Reduced p.p.-rings without identity*, International Journal of Mathematics and Mathematical Science, Vol.2006, 2006, Art. ID 93890, 7 pp.

[9] X.J. Guo and K.P. Shum, *On p.p. rings which are reduced*, International Journal of Mathematics and Mathematical Science, Vol.2006, 2006, Art. ID 34694, 5 pp.

[10] Y. Hirano, M. Hongan and M. Ôhori, *On right PP rings*, Math. J. Okayama Univ., Vol.24, 1982, pp.99-109.

[11] C.Y. Hong, N.K. Kim and T.K. Kwak, *Ore extensions of Baer and P.P.-rings*, J. Pure and Appl. Algebra, Vol.151, 2000, pp.215-226.

[12] S. J∅ndrup, *P.P. rings and finitely generated flat ideals,* Proc. Amer. Math. Soc., Vol.28, 1971, pp.431-435.

[13] Z.K. Liu and J. Ahsan, *P.P.-rings of generalized power series,* Acta Math. Sinica (new ser.), Vol.16, 2000, pp.573-578.

[14] S. Maeda, *On a ring whose principal right ideals generated by idempotents,* J. Sci. Hiroshima Univ. (Ser. A), Vol.24, 1960, pp.509-525.

[15] L.W. Small, *Semiherditary rings*, Bull. Amer. Math. Soc., Vol.73, 1967, pp.656-658.

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Received by the editors: 01.05.2008
and in final form 01.05.2008.