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RESEARCH ARTICLE

On closed rational functions in several variables

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Dedicated to V.I. Sushchansky on the occasion of his 60th birthday

ABSTRACT. Let $\mathbb{K} = \bar{\mathbb{K}}$ be a field of characteristic zero. An element $\varphi \in \mathbb{K}(x_1, \ldots, x_n)$ is called a closed rational function if the subfield $\mathbb{K}(\varphi)$ is algebraically closed in the field $\mathbb{K}(x_1, \ldots, x_n)$. We prove that a rational function $\varphi = f/g$ is closed if f and g are algebraically independent and at least one of them is irreducible. We also show that a rational function $\varphi = f/g$ is closed if and only if the pencil $\alpha f + \beta g$ contains only finitely many reducible hypersurfaces. Some sufficient conditions for a polynomial to be irreducible are given.

Introduction

Closed polynomials, i.e., polynomials $f \in \mathbb{K}[x_1, \ldots, x_n]$ such that the subalgebra $\mathbb{K}[f]$ is integrally closed in $\mathbb{K}[x_1, \ldots, x_n]$, were studied by many authors (see, for example, [5], [10], [4], [1]). A rational analogue of a closed polynomial is a rational function φ such that the subfield $\mathbb{K}(\varphi)$ is algebraically closed in the field $\mathbb{K}(x_1, \ldots, x_n)$, such a rational function will be called a closed one. Although there are algorithms to determine whether a given rational function is closed (see, for example, [7]) it is interesting to study closed rational functions more detailed.

We give the following sufficient condition for a rational function to be closed. Let $\varphi = f/g \in \mathbb{K}(x_1, \ldots, x_n)$, f and g are coprime, algebraically independent and at least one of polynomials f and g is irreducible. Then φ is a closed rational function (Theorem 1).

Using some results of of J. M. Ollagnier [7] about Darboux polynomials we prove that a rational function $\varphi = f/g \in \mathbb{K}(x_1, \ldots, x_n)\backslash\mathbb{K}$ is closed

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if and only if the pencil $\alpha f + \beta g$ of hypersurfaces contains only finitely many reducible hypersurfaces (Theorem 2). We also study products of irreducible polynomials.

Notations in the paper are standard. For a rational function $F(t) \in \mathbb{K}(t)$ of the form $F(t) = \frac{P(t)}{Q(t)}$ with coprime polynomials P and Q the degree is $\deg F = \max(\deg P, \deg Q)$. The ground field $\mathbb{K}$ is algebraically closed of characteristic 0.

1. Closed rational functions in several variables

Lemma 1. *For a rational functions $\varphi, \psi \in \mathbb{K}(x_1, \dots, x_n)\backslash\mathbb{K}$ the following conditions are equivalent.*

1) φ and ψ are algebraically dependent over $\mathbb{K}$;

2) the rank of Jacobi matrix $J(\varphi, \psi) = \begin{pmatrix} \frac{\partial\varphi}{\partial x_1} & \dots & \frac{\partial\varphi}{\partial x_n} \\ \frac{\partial\psi}{\partial x_1} & \dots & \frac{\partial\psi}{\partial x_n} \end{pmatrix}$ is equal to 1;

3) for differentials $\mathrm{d}\,\varphi$ and $\mathrm{d}\,\psi$ of functions φ and ψ respectively it holds $\mathrm{d}\,\varphi \wedge \mathrm{d}\,\psi = 0$;

4) there exists $h \in \mathbb{K}(x_1, \dots, x_n)$ such that $\varphi = F(h)$ and $\psi = G(h)$ for some $F(t), G(t) \in \mathbb{K}(t)$.

Proof. The equivalence of 1) and 2) follows from [3], Ch. III, §7, Th. III. The equivalence of 2) and 3) is obvious. Since 2) clearly follows from 4), it remains to show that 1) implies 4). Let φ and ψ be algebraically dependent. Then obviously $\mathrm{tr.deg}_{\mathbb{K}}\, \mathbb{K}(\varphi, \psi) = 1$. By Theorem of Gordan (see for example [9], p.15) $\mathbb{K}(\varphi, \psi) = \mathbb{K}(h)$ for some rational function h and therefore $\varphi = F(h)$ and $\psi = G(h)$ for some $F(t), G(t) \in \mathbb{K}(t)$. $\square$

Definition 1. *We call a rational function $\varphi \in \mathbb{K}(x_1, \dots, x_n) \setminus \mathbb{K}$ closed if the subfield $\mathbb{K}(\varphi)$ is algebraically closed in $\mathbb{K}(x_1, \dots, x_n)$.*

A rational function $\tilde{\psi}$ is called generative for a rational function ψ if $\tilde{\psi}$ is closed and $\psi \in \mathbb{K}(\tilde{\psi})$.

Lemma 2. *1) For a rational function $\varphi \in \mathbb{K}(x_1, \dots, x_n)\backslash\mathbb{K}$ the following conditions are equivalent.*

a) φ is closed;

b) $\mathbb{K}(\varphi)$ is a maximal element in the partially ordered (by inclusion) set of subfields of $\mathbb{K}(x_1, \dots, x_n)$ of the form $\mathbb{K}(\psi)$, $\psi \in \mathbb{K}(x_1, \dots, x_n) \setminus \mathbb{K}$.

c) φ is non-composite rational function, i.e., from the equality $\varphi = F(\psi)$, for some rational functions $\psi \in \mathbb{K}(x_1, \dots, x_n)\backslash\mathbb{K}$ and $F(t) \in \mathbb{K}(t)$, it follows that $\deg F = 1$.

2) For every rational function $\varphi \in \mathbb{K}(x_1, \dots, x_n) \setminus \mathbb{K}$ there exists a generative rational function $\tilde{\varphi}$. If $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are two generative rational *functions for* φ*, then* $\tilde{\varphi}_2 = \frac{a\tilde{\varphi}_1+b}{c\tilde{\varphi}_1+d}$ *for some* $a, b, c, d \in \mathbb{K}$ *such that* $ad-bc \neq 0$.

Proof. 1) a)$\Rightarrow$ b). Suppose that the rational function φ is closed and $\mathbb{K}(\varphi) \subseteq \mathbb{K}(\psi)$ for some $\psi \in \mathbb{K}(x_1, \dots, x_n)$. The element ψ is algebraic over over $\mathbb{K}(\varphi)$ and therefore by the definition of closed rational functions we have $\psi \in \mathbb{K}(\varphi)$. Thus $\mathbb{K}(\varphi)$ is a maximal element in the set of all one-generated subfields of $\mathbb{K}(x_1, \dots, x_n)$.

b)$\Rightarrow$ a). If $\mathbb{K}(\varphi)$ is a maximal one-generated subfield of $\mathbb{K}(x_1, \dots, x_n)$, then $\mathbb{K}(\varphi)$ is algebraically closed in $\mathbb{K}(x_1, \dots, x_n)$. Indeed, if f is algebraic over $\mathbb{K}(\varphi)$, then $\operatorname{tr.deg}_{\mathbb{K}}(\varphi, f) = 1$ and by Theorem of Gordan $\mathbb{K}(\varphi, f) = \mathbb{K}(\psi)$ for some rational function ψ. But then $\mathbb{K}(\psi) = \mathbb{K}(\varphi)$ and $f \in \mathbb{K}(\varphi)$.

The equivalence of b) and c) is obvious.

2) The subfield $\mathbb{K}(\varphi)$ is contained in some maximal one-generated subfield $\mathbb{K}(\tilde{\varphi})$, which is algebraically closed in $\mathbb{K}(x_1, \dots, x_n)$ by the part 1) of this Lemma. Therefore $\tilde{\varphi}$ is a generative rational function for φ.

Let $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ be two generative rational functions for φ. Then $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are algebraic over the field $\mathbb{K}(\varphi)$ and therefore $\operatorname{tr.deg}_{\mathbb{K}} \mathbb{K}(\varphi, \tilde{\varphi}_1, \tilde{\varphi}_2) = 1$. In particular, then the rational functions $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are algebraically dependent.

By Lemma 1 one obtains $\tilde{\varphi}_1 \in \mathbb{K}(\psi)$, $\tilde{\varphi}_2 \in \mathbb{K}(\psi)$ for some rational function ψ. Since both $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are closed, we get $\mathbb{K}(\tilde{\varphi}_1) = \mathbb{K}(\psi) = \mathbb{K}(\tilde{\varphi}_2)$. But there exists a fractional rational transformation θ of the field $\mathbb{K}(\psi)$ such that $\theta(\tilde{\varphi}_2) = \tilde{\varphi}_1$. Therefore, $\tilde{\varphi}_2 = \frac{a\tilde{\varphi}_1+b}{c\tilde{\varphi}_1+d}$, for some $a, b, c, d \in \mathbb{K}$, $ad - bc \neq 0$. □

Remark 1. Note that algebraically dependent rational functions have the same set of generative functions. This follows from Lemma 1 and Lemma 2.

Remark 2. Let $f \in \mathbb{K}[x_1, \dots, x_n] \setminus \mathbb{K}$. By Lemma 3 from [1], the subfield $\mathbb{K}(f)$ is algebraically closed if and only if the polynomial f is closed. So, the polynomial f is closed if and only if f is closed as a rational function.

Theorem 1. *Let polynomials* $f, g \in \mathbb{K}[x_1, \dots, x_n]$ *be coprime and algebraically independent. If at least one of them is irreducible, then the rational function* $\varphi = \frac{f}{g}$ *is closed.*

Proof. Without loss of generality we can assume that f is irreducible. By Lemma 2 there exists a generative rational function $\psi = \frac{p}{q}$ for φ, where p and q are coprime polynomials. Then $\varphi = \frac{P(\psi)}{Q(\psi)}$ for some coprime polynomials $P(t), Q(t) \in \mathbb{K}[t]$.

Let $P(t) = a_0(t-\lambda_1)\dots(t-\lambda_m)$ and $Q(t) = b_0(t-\mu_1)\dots(t-\mu_l)$, $\lambda_i, \mu_j \in \mathbb{K}$ be the decompositions of $P(t)$ and $Q(t)$ into irreducible factors. Then

$$\varphi = \frac{f}{g} = \frac{a_0(\frac{p}{q}-\lambda_1)\dots(\frac{p}{q}-\lambda_m)}{b_0(\frac{p}{q}-\mu_1)\dots(\frac{p}{q}-\mu_l)} = \frac{a_0(p-\lambda_1 q)\dots(p-\lambda_m q)q^{l-m}}{b_0(p-\mu_1 q)\dots(p-\mu_l q)}.$$

and we obtain

$$(*) \qquad b_0 f(p-\mu_1 q)\dots(p-\mu_l q) = a_0 g(p-\lambda_1 q)\dots(p-\lambda_m q)q^{l-m}.$$

Note, as $\lambda_i \neq \mu_j$, the polynomials $p-\lambda_i q$ and $p-\mu_j q$ are coprime for all $i=\overline{1,l}$ and $j=\overline{1,m}$. Moreover, since p and q are coprime, it is clear that q is coprime with polynomials of the form $p+\alpha q$, $\alpha \in \mathbb{K}$.

Note also that $p-\beta q \notin \mathbb{K}$. Indeed, if $p-\beta q = \xi \in \mathbb{K}$ for some $\beta, \xi \in \mathbb{K}$, then $p = \xi + \beta q$ and

$$\varphi = \frac{f}{g} = \frac{a_0(\xi+(\beta-\lambda_1)q)\dots(\xi+(\beta-\lambda_m)q)q^{l-m}}{b_0(\xi+(\beta-\mu_1 q)\dots(\xi+(\beta-\mu_l)q)},$$

which means that f and g are algebraically dependent, which contradicts our assumptions.

So from $(*)$ we conclude that f is divisible by all polynomials $(p-\lambda_i q)$. Since f is irreducible, taking into account the above considerations we conclude that $m=1$ and $f = a(p-\lambda_1 q)$, $a \in \mathbb{K}^*$. Therefore, from $(*)$ we obtain

$$b_0 a(p-\lambda_1 q)(p-\mu_1 q)\dots(p-\mu_l q) = a_0 g(p-\lambda_1 q)q^{l-1}$$

and after reduction

$$b_0 a(p-\mu_1 q)\dots(p-\mu_l q) = a_0 g q^{l-1}.$$

Since q is coprime with $(p-\mu_j q)$, we get $l=1$ and finally $a_0 g = b_0 a(p-\mu_1 q)$. We obtain

$$\varphi = \frac{f}{g} = \frac{a_0(p-\lambda_1 q)}{b_0(p-\mu_1 q)} = \frac{a_0(\frac{p}{q}-\lambda_1)}{b_0(\frac{p}{q}-\mu_1)} = \frac{a_0(\psi-\lambda_1)}{b_0(\psi-\mu_1)}.$$

One concludes that $\mathbb{K}(\varphi) = \mathbb{K}(\psi)$, which means that φ is a closed rational function. □

2. Rational functions and pencils of hypersurfaces

In this section we give a characterization of closed rational functions. While proving Theorem 2 we use an approach from the paper of J. M. Ollagnier [7] connected with Darboux polynomials. Recall some notions and terminology (see also [6], pp.22–24). If δ is a derivation of the polynomial ring $\mathbb{K}[x_1, \dots, x_n]$, then a polynomial f is called a Darboux polynomial for δ if $\delta(f) = \lambda f$ for some polynomial λ (not necessarily $\lambda \in \mathbb{K}$). The polynomial λ is called the cofactor for δ corresponding to the Darboux polynomial f (so, f is a polynomial eigenfunction for δ and λ is the corresponding eigenvalue).

Further, for a rational function $\varphi = \frac{f}{g} \in \mathbb{K}(x_1, \dots, x_n) \setminus \mathbb{K}$ one can define a (vector) derivation $\delta_\varphi = gdf - fdg : \mathbb{K}[x_1, \dots, x_n] \to \Lambda^2 \mathbb{K}[x_1, \dots, x_n]$ by the rule $\delta_\varphi(h) = dh \wedge (gdf - fdg)$. For such a derivation δ_φ a polynomial h is called a Darboux polynomial if all coefficients of the 2−form $dh \wedge (gdf - fdg)$ are divisible by the polynomial h, i.e., $dh \wedge (gdf - fdg) = h \cdot \lambda$ for some 2−form λ, which is called a cofactor for the derivation δ_φ. Note that every divisor of the Darboux polynomial h is also a Darboux polynomial for δ_φ (see, for example, [6], p.23). It is easy to see that the polynomial $\alpha f + \beta g$ is a Darboux polynomial for the derivation δ_φ and therefore every divisor of the polynomial $\alpha f + \beta g$ is a Darboux polynomial of the derivation $\delta_\varphi = gdf - fdg$.

Theorem 2. *Let polynomials $f, g \in \mathbb{K}[x_1, \dots, x_n]$ be coprime and let at least one of them be a non-constant polynomial. Then the rational function $\varphi = \frac{f}{g}$ is closed if and only if all but finitely many hypersurfaces in the pencil $\alpha f + \beta g$ are irreducible.*

Proof. Let $\varphi = \frac{f}{g}$ be closed. Suppose that the pencil $\alpha f + \beta g$ contains infinitely many reducible hypersurfaces. Let $\{\alpha_i f + \beta_i g\}_{i \in \mathbb{N}}$, $(\alpha_i : \beta_i) \neq (\alpha_j : \beta_j)$ for $i \neq j$ as points of $\mathbb{P}^1$, be an infinite sequence of (different) reducible hypersurfaces. For each i take one irreducible factor h_i of $\alpha_i f + \beta_i g$.

By the above remark, all polynomials h_i are Darboux polynomials for δ_φ and $\deg h_i < \deg \varphi$. By Corollary 5 from [7] there exist finitely many cofactors of δ_φ that correspond to Darboux polynomials h_i (degrees of h_i are bounded). Therefore, there exist polynomials h_i and h_j such that $\delta_\varphi(h_i) = \lambda h_i$ and $\delta_\varphi(h_j) = \lambda h_j$ for some cofactor $\lambda \in \bigwedge^2 \mathbb{K}[x_1, \dots, x_n]$. This implies $\delta_\varphi(\frac{h_i}{h_j}) = 0$ and thus $\mathrm{d}(\frac{f}{g}) \wedge \mathrm{d}(\frac{h_i}{h_j}) = \frac{1}{g^2} \delta_\varphi(\frac{h_i}{h_j}) = 0$ (see [7]). Then by Lemma 1, the rational functions $\varphi = \frac{f}{g}$ and $\frac{h_i}{h_j}$ are algebraically dependent. As φ is closed, $\frac{h_i}{h_j} = F(\varphi)$ for some $F(t) \in \mathbb{K}(t)$ and therefore $\deg \frac{h_i}{h_j} = \deg F \deg \varphi$ (see for example [7]). But this is impossible since

$\deg \frac{h_i}{h_j} < \deg \varphi$. Therefore, all but finitely many hypersurfaces in $\alpha f + \beta g$ are irreducible.

Let now $\alpha_0 f + \beta_0 g$ be an irreducible hypersurface from the pencil $\alpha f + \beta g$. Consider the case when f and g are algebraically independent. One can assume without loss of generality $\alpha_0 \neq 0$. Then $\alpha_0 f + \beta_0 g$ and g are algebraically independent as well. (If $\alpha_0 = 0$, then $\beta_0 \neq 0$ and polynomials f and $\alpha_0 f + \beta_0 g$ are algebraically independent). Therefore, since $\alpha_0 f + \beta_0 g$ and g are coprime, by Theorem 1 the rational function $\psi = \frac{\alpha_0 f + \beta_0 g}{g}$ is closed. Then obviously $\varphi = \frac{f}{g} = \alpha_0^{-1}(\psi - \beta_0)$, which proves that φ is a closed rational function.

Let now f and g be algebraically dependent. Then $f = F(h)$ and $g = G(h)$ for a common generative polynomial function h and polynomials $F(t), G(t) \in \mathbb{K}[t]$ (see Remark 2). Let $(1 : \beta_1) \neq (1 : \beta_2)$ be two different points in $\mathbb{P}^1$ such that $f + \beta_i g = F(h) + \beta_i G(h)$ is irreducible for $i \in \{1, 2\}$. In particular this means that $\deg(F(t) + \beta_i G(t)) = 1$, i.e., $F(t) + \beta_i G(t) = a_i t + b_i$, $a_i, b_i \in \mathbb{K}$, $a_i \neq 0$. Then since $\beta_1 \neq \beta_2$, we conclude that $F(t) = at + b$ and $G(t) = ct + d$ for some $a, b, c, d \in \mathbb{K}$. So $\varphi = \frac{f}{g} = \frac{ah+b}{ch+d}$. As at least one of f and g is non-constant and since f and g are coprime, we conclude that $\mathbb{K}(\varphi) = \mathbb{K}(h)$. Therefore, since $\mathbb{K}(h)$ is an algebraically closed subfield of the field $\mathbb{K}(x_1, \dots, x_n)$, $\varphi = \frac{f}{g}$ is a closed rational function. □

Remark 3. Note, in order to show that $\varphi = \frac{f}{g}$ is closed in Theorem 2 it is enough to have two different irreducible hypersurfaces in the pencil $\alpha f + \beta g$. One irreducible hypersurface $\alpha_0 f + \beta_0 g$ is enough provided f and g are algebraically independent.

Remark 4. We also reproved a weak version (we do not give any estimation) of the next result of W. Ruppert (see [8], Satz 6).

If f and g are algebraically independent polynomials and the pencil $\alpha f + \beta g$ contains at least one irreducible hypersurface, then all but finitely many hypersurfaces in $\alpha f + \beta g$ are irreducible.

Remark 5. If a polynomial $f \in \mathbb{K}[x_1, \dots, x_n]$ is non-constant then by Theorem 2 and Remark 2 f is closed if and only if for all but finitely many $\lambda \in \mathbb{K}$ the polynomial $f + \lambda$ is irreducible. This result is well-known (it can be proved by using the first Bertini theorem), see, for example, [9], Corollary 3.3.1.

Using Remark 5 one can prove that any non-constant polynomial $f \in \mathbb{K}[x_1, \dots, x_n] \setminus \mathbb{K}$ can be written in the form $f = F(h)$ for some polynomial $F(t) \in \mathbb{K}[t]$ and irreducible polynomial h. Similar statement holds for rational functions.

Corollary 1. *A rational function* $\frac{f}{g} \in \mathbb{K}(x_1, \ldots, x_n) \setminus \mathbb{K}$ *can be written in the form* $\frac{f}{g} = F(\varphi)$, $F(t) \in \mathbb{K}(t)$, *for some rational function* $\varphi = \frac{p}{q}$ *such that polynomials* p *and* q *are irreducible.*

Proof. Let $\frac{p_1}{q_1}$ be a generative function for $\frac{f}{g}$. As $\frac{p_1}{q_1}$ is closed, by Theorem 2 the pencil $\alpha p_1 + \beta q_1$ contains two different irreducible hypersurfaces $p = \alpha_1 p_1 + \beta_1 q_1$ and $q = \alpha_2 p_1 + \beta_2 q_1$, i.e., with $(\alpha_1 : \beta_1) \neq (\alpha_2 : \beta_2)$. Since the pencils $\alpha p + \beta q$ and $\alpha p_1 + \beta q_1$ are equal, and since in the pencil $\alpha p_1 + \beta q_1$ all but finitely many hypersurfaces are irreducible, we conclude that $\frac{p}{q}$ is closed and is a generative function for $\frac{f}{g}$. □

Remark 6. Under conditions of Corollary 1 polynomials p and q can be chosen of the same degree.

Corollary 2. *Let* $\mathbb{K} \subseteq L \subseteq \mathbb{K}(x_1, \ldots, x_n)$ *be an algebraically closed subfield in* $\mathbb{K}(x_1, \ldots, x_n)$. *Then it is possible to choose generators of* L *in the form* $\frac{p_1}{q_1}, \ldots, \frac{p_m}{q_m}$, *where* p_i *and* q_i *are irreducible polynomials.*

Theorem 3. *Let polynomials* $f, g \in \mathbb{K}[x_1, \ldots, x_n]$ *be coprime and algebraically independent. Then the rational function* f/g *is not closed if and only if there exist algebraically independent irreducible polynomials* p *and* q *and a positive integer* $k \geqslant 2$ *such that* $f = (\alpha_1 p + \beta_1 q) \ldots (\alpha_k p + \beta_k q)$ *and* $g = (\gamma_1 p + \delta_1 q) \ldots (\gamma_k p + \delta_k q)$ *for some* $(\alpha_i : \beta_i), (\gamma_j : \delta_j) \in \mathbb{P}^1$, *with* $(\alpha_i : \beta_i) \neq (\gamma_j : \delta_j)$, $i, j = \overline{1, k}$.

Proof. Suppose $\frac{f}{g}$ is not closed. Take its generative function $\frac{p}{q}$ with irreducible polynomials p and q (this is possible by Corollary 1). Then $\frac{f}{g} = F(\frac{p}{q})$ for some rational function $F(t) \in \mathbb{K}(t)$ with $\deg F(t) = k \geqslant 2$. Note that the polynomials p and q are algebraically independent because in other case the polynomials f and g were algebraically dependent which contradicts to our assumptions. Write

$$F(t) = \frac{a_0(t - \lambda_1) \ldots (t - \lambda_s)}{b_0(t - \mu_1) \ldots (t - \mu_r)}$$

with $\lambda_i \neq \mu_j$, i.e., with coprime nominator and denominator. It is clear that $k = \deg F(t) = \max\{s, r\}$. After substitution of t by $\frac{p}{q}$ we obtain

$$\frac{f}{g} = \frac{a_0(p - \lambda_1 q) \ldots (p - \lambda_s q) q^{r-s}}{b_0(p - \mu_1 q) \ldots (p - \mu_r q)}.$$

Put $(\alpha_i : \beta_i) = (1 : -\lambda_i)$ for $i = \overline{1, s}$, $(\gamma_j : \delta_j) = (1 : -\mu_j)$ for $j = \overline{1, r}$. If $r \leqslant s$, then put $(\gamma_j : \delta_j) = (0 : 1)$ for $j = r + 1, \ldots, s$. If $r > s$, then put $(\alpha_i : \beta_i) = (0 : 1)$ for $i = s + 1, \ldots, r$. We obtained

$$\frac{f}{g} = \frac{a_0(\alpha_1 p + \beta_1 q) \ldots (\alpha_k p + \beta_k q)}{b_0(\gamma_1 p + \delta_1 q) \ldots (\gamma_k p + \delta_k q)},$$

which means that up to multiplication by a non-zero constant $f = (\alpha_1 p + \beta_1 q)\dots(\alpha_k p + \beta_k q)$ and $g = (\gamma_1 p + \delta_1 q)\dots(\gamma_k p + \delta_k q)$.

Suppose now that f and g have the form as in the conditions of this Theorem. Let us show that the rational function $\frac{f}{g}$ is not closed. As $f = (\alpha_1 p + \beta_1 q)\dots(\alpha_k p + \beta_k q)$ and $g = (\gamma_1 p + \delta_1 q)\dots(\gamma_k p + \delta_k q)$, one has

$$\frac{f}{g} = \frac{(\alpha_1 p + \beta_1 q)\dots(\alpha_k p + \beta_k q)}{(\gamma_1 p + \delta_1 q)\dots(\gamma_k p + \delta_k q)} = \frac{(\alpha_1 \frac{p}{q} + \beta_1)\dots(\alpha_k \frac{p}{q} + \beta_k)}{(\gamma_1 \frac{p}{q} + \delta_1)\dots(\gamma_k \frac{p}{q} + \delta_k)},$$

i.e., $\frac{f}{g} = F(\frac{p}{q})$ for the rational function $F(t) = \frac{(\alpha_1 t+\beta_1)\dots(\alpha_k t+\beta_k)}{(\gamma_1 t+\delta_1)\dots(\gamma_k t+\delta_k)}$. Since $(\alpha_i : \beta_i) \neq (\gamma_j : \delta_j)$, $i, j = \overline{1,k}$, we conclude that $\deg F(t) \geqslant 2$, which means that $\frac{f}{g}$ is not closed (equivalently, by Theorem 2, the pencil $\alpha f + \beta g$ contains infinitely many reducible hypersurfaces). □

Example 1. Let p and q be irreducible algebraically independent polynomials in $\mathbb{K}[x_1, \dots, x_n]$, $n \geqslant 2$. Then $\varphi = \frac{p^l}{q^m}$ is a closed rational function for coprime l and m.

Indeed, suppose the converse holds. Then by Theorem 3 there exists irreducible polynomials p_1 and q_1, an integer $k \geqslant 2$ such that

$$\frac{p^l}{q^m} = \frac{(\alpha_1 p_1 + \beta_1 q_1)\dots(\alpha_k p_1 + \beta_k q_1)}{(\gamma_1 p_1 + \delta_1 q_1)\dots(\gamma_k p_1 + \delta_k q_1)}, \quad (\alpha_i : \beta_i) \neq (\gamma_j : \delta_j),\ i, j = \overline{1,k}.$$

Since $\alpha_i p_1 + \beta_i q_1$ and $\gamma_j p_1 + \delta_j q_1$ are coprime for all i and j, it follows that $p^l = (\alpha_1 p_1 + \beta_1 q_1)\dots(\alpha_k p_1 + \beta_k q_1)$ and $q^m = (\gamma_1 p_1 + \delta_1 q_1)\dots(\gamma_k p_1 + \delta_k q_1)$. Since p and q are algebraically independent, as in the proof of Theorem 1 we conclude that $\alpha p_1 + \beta q_1 \neq \mathbb{K}$ for all $(\alpha : \beta) \in \mathbb{P}^1$. Therefore, $(\alpha_1 : \beta_1) = \dots = (\alpha_k : \beta_k)$, $(\gamma_1 : \delta_1) = \dots = (\gamma_k : \delta_k)$, and

$$p^l = a_0(\alpha_1 p_1 + \beta_1 q_1)^k, \quad q^m = b_0(\gamma_1 p_1 + \delta_1 q_1)^k$$

for some $a_0, b_0 \in \mathbb{K}^*$. Since p and q are irreducible, we obtain, up to multiplication by a non-zero constant, $\alpha_1 p_1 + \beta_1 q_1 = p^{l'}$ and $\gamma_1 p_1 + \delta_1 q_1 = q^{m'}$, i.e., $k \geqslant 2$ divides both l and m. This is impossible, since l and m are coprime. We obtained a contradiction, which proves that $\frac{p^l}{q^m}$ is a closed rational function.

3. Products of irreducible polynomials

Theorem 4. *Let $p_1, \dots, p_k \in \mathbb{K}[x_1, \dots, x_n]$ be irreducible algebraically independent polynomials. If* $\gcd(m_1, m_2, \dots m_k) = 1$ *then the polynomial*

$$p_1^{m_1} p_2^{m_2} \dots p_k^{m_k} + \lambda$$

is irreducible for all but finitely many $\lambda \in \mathbb{K}$.

Proof. Show at first that the polynomial $p_1^{m_1}p_2^{m_2}\dots p_k^{m_k}$ is closed. Suppose to the contrary it is not closed and let $p_1^{m_1}p_2^{m_2}\dots p_k^{m_k} = F(h)$ for some closed polynomial h and $F(t) \in \mathbb{K}[t]$, $\deg F(t) \geqslant 2$. Let $F(t) = \alpha(t-\mu_1)\dots(t-\mu_s)$ be the decomposition of $F(t)$ into linear factors. Then

$$p_1^{m_1}p_2^{m_2}\dots p_k^{m_k} = \alpha(h-\mu_1)\dots(h-\mu_s), \quad \mu_i \in \mathbb{K}, \quad \alpha \in \mathbb{K}^*.$$

Since the polynomials $h-\mu_i$ are closed and since we assumed that the polynomial $p_1^{m_1}p_2^{m_2}\dots p_k^{m_k}$ is not closed, one concludes that $s \geqslant 2$. Suppose there exists $\mu_i \neq \mu_j$, assume without loss of generality $\mu_1 \neq \mu_2$. As all p_i are irreducible, it is clear that $h-\mu_1 = \alpha_1 p_{i_1}^{s_1}\dots p_{i_m}^{s_m}$ and $h-\mu_2 = \alpha_2 p_{j_1}^{t_1}\dots p_{j_r}^{t_r}$ for $p_{i_1},\dots,p_{i_m},p_{j_1},\dots,p_{j_r} \in \{p_1,\dots,p_k\}$. Since $\mu_1 \neq \mu_2$, the polynomials $h-\mu_1$ and $h-\mu_2$ are coprime. Therefore, the sets $\{p_{i_1},\dots,p_{i_m}\}$ and $\{p_{j_1},\dots,p_{j_r}\}$ are disjoint. From $(h-\mu_1)-(h-\mu_2)+(\mu_1-\mu_2)=0$ it follows that

$$\alpha_1 p_{i_1}^{s_1}\dots p_{i_m}^{s_m} - \alpha_2 p_{j_1}^{t_1}\dots p_{j_r}^{t_r} + (\mu_1-\mu_2) = 0,$$

which means that the set $\{p_1,\dots,p_k\}$ is algebraically dependent. We obtained a contradiction. Therefore, $\mu_1 = \dots = \mu_s$ and $p_1^{m_1}\dots p_k^{m_k} = \alpha(h-\mu_1)^s$, $s \geqslant 2$. From the unique factorization of the polynomial $p_1^{m_1}\dots p_k^{m_k}$ it follows that $s|m_1,\dots,s|m_k$ which is impossible by our restriction on numbers $m_1,\dots,m_k$. This contradiction proves that the polynomial $p_1^{m_1}\dots p_k^{m_k}$ is closed. Therefore, by Remark 5 the polynomial $p_1^{m_1}\dots p_k^{m_k}+\lambda$ is irreducible for all but finitely many $\lambda \in \mathbb{K}$. □

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