Algebra and Discrete Mathematics
Number 1. **(2007).** pp. 13 – 28

RESEARCH ARTICLE



# On Frobenius full matrix algebras with structure systems

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Abstract. Let $n \geq 2$ be an integer. In [5] and [6], an $n \times n$ $\mathbb{A}$-full matrix algebra over a field $K$ is defined to be the set $\mathbb{M}_n(K)$ of all square $n \times n$ matrices with coefficients in $K$ equipped with a multiplication defined by a structure system $\mathbb{A}$, that is, an $n$-tuple of $n \times n$ matrices with certain properties. In [5] and [6], mainly $\mathbb{A}$-full matrix algebras having $(0, 1)$-structure systems are studied, that is, the structure systems $\mathbb{A}$ such that all entries are 0 or 1. In the present paper we study $\mathbb{A}$-full matrix algebras having non $(0, 1)$-structure systems. In particular, we study the Frobenius $\mathbb{A}$-full matrix algebras. Several infinite families of such algebras with nice properties are constructed in Section 4.

## 1. Introduction

Throughout this paper we freely use the rings, modules, and representation theory terminology introduced in [1], [2], [4], [9], [11], and [12]. In particular, given a finite dimensional algebra $R$ over a field $K$, we denote by $\operatorname{mod} R$ the category of all finite dimensional unitary right $R$-modules. Given a module $M$ in $\operatorname{mod} R$, we denote by $\operatorname{soc} M$ the socle of $M$.

Let $K$ be a field and $n \geq 2$ an integer. Let $\mathbb{A} = [A_1, \ldots, A_n] = [a_{ij}^{(k)}]_{i,j,k}$ be an $n$-tuple of $n \times n$ matrices $A_k = (a_{ij}^{(k)}) \in \mathbb{M}_n(K)$ $(1 \leq k \leq n)$ satisfying the following three conditions:

(A1) $a_{ij}^{(k)} a_{il}^{(j)} = a_{il}^{(k)} a_{kl}^{(j)}$, for all $i, j, k, l \in \{1, \ldots, n\}$,

(A2) $a_{kj}^{(k)} = a_{ik}^{(k)} = 1$, for all $i, j, k \in \{1, \ldots, n\}$, and

**2000 Mathematics Subject Classification:** *16G10, 16G30, 16G60.*
**Key words and phrases:** *Frobenius algebra, quiver, module, socle, tame representation type.*

(A3) $a_{ii}^{(k)} = 0$, for all $i, k \in \{1, \ldots, n\}$ such that $i \neq k$.

We denote by

$$R_{\mathbb{A}} = \bigoplus_{i,j=1}^{n} K u_{ij} \tag{1.1}$$

a $K$-vector space, with basis $\{u_{ij} \mid 1 \leq i, j \leq n\}$, equipped with a multiplication (depending on $\mathbb{A}$) defined by the formula

$$u_{ik}u_{lj} := \begin{cases} a_{ij}^{(k)} u_{ij}, & \text{if } \ k = l, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check that $R_{\mathbb{A}}$ is an associative, basic $K$-algebra $u_{11}, \ldots, u_{nn}$ are orthogonal primitive idempotents of $R_{\mathbb{A}}$ and $1 = u_{11} + \cdots + u_{nn}$ is an identity element of $R_{\mathbb{A}}$, see [5, Proposition 1.1]. We call $R_{\mathbb{A}}$ an $\mathbb{A}$-*full matrix algebra* and $\mathbb{A}$ a *structure system* of $R_{\mathbb{A}}$.

The reader is referred to the recent paper [7] for a degeneration-like approach to the full matrix algebras $R_{\mathbb{A}}$ with structure systems.

Since $u_{ii}R_{\mathbb{A}}u_{jj} \neq 0$, for all $1 \leq i, j \leq n$, then the $K$-algebra $R_{\mathbb{A}}$ is *connected*, that is, $R_{\mathbb{A}}$ can not be decomposed into a product of two subalgebras. Note also that the Jacobson radical $J(R_{\mathbb{A}})$ of $R_{\mathbb{A}}$ has the form

$$J(R_{\mathbb{A}}) = \bigoplus_{i \neq j} u_{ij}K. \tag{1.2}$$

If $V$ is a simple right $R_{\mathbb{A}}$-module, then $Vu_{ii} \neq 0$, for some $1 \leq i \leq n$, and $V \cong u_{ii}R_{\mathbb{A}}/u_{ii}J(R_{\mathbb{A}})$ . Therefore the $R_{\mathbb{A}}$-modules

$$u_{11}R_{\mathbb{A}}/u_{11}J(R_{\mathbb{A}}), \ldots, u_{nn}R_{\mathbb{A}}/u_{nn}J(R_{\mathbb{A}})$$

are the representatives of all pairwise non-isomorphic simple right $R_{\mathbb{A}}$-modules. Note that $\dim_K V = 1$, for any simple right $R_{\mathbb{A}}$-module $V$.

Let $R_{\mathbb{A}}$ be an $\mathbb{A}$-full matrix algebra (1.1) and let $M$ be a right $R_{\mathbb{A}}$-module in $\operatorname{mod} R_{\mathbb{A}}$. The *dimension vector* of $M$ (or the dimension type of $M$) is defined to be the $n$-tuple

$$\underline{\dim}\, M = (d_1, \ldots, d_n) \in \mathbb{Z}^n = K_0(R_{\mathbb{A}}) \tag{1.3}$$

of integers $d_i = \dim_K M u_{ii}$, with $1 \leq i \leq n$, see [1] and [5].

## 2. When $\mathbb{A}$-full matrix algebras are isomorphic?

In this section, we give a criterion for two $\mathbb{A}$-full matrix algebras $R_{\mathbb{A}}$ and $R_{\mathbb{B}}$ to be isomorphic. Moreover, we give a list of the representatives of all non-isomorphic $3 \times 3$ $\mathbb{A}$-full matrix algebras.

The isomorphism problem of $\mathbb{A}$-full matrix algebras is also studied in [7] in terms of an action

$$* : \mathbb{G}_n(K) \times \mathbb{ST}_n(K) \ \rightarrow \ \mathbb{ST}_n(K)$$

of an algebraic group $\mathbb{G}_n(K)$ (containing the symmetric group $S_n$) on the algebraic $K$-variety $\mathbb{ST}_n(K)$ of the structure systems $\mathbb{A}$.

**Proposition 2.1.** *Let* $R_{\mathbb{A}} = \bigoplus_{i,j=1}^{n} Ku_{ij}$ *and* $R_{\mathbb{B}} = \bigoplus_{i,j=1}^{n} Kv_{ij}$ *be full matrix algebras with structure systems*

$$\mathbb{A} = [A_1, \ldots, A_n] = [a_{ij}^{(k)}]_{i,j,k} \quad \text{and} \quad \mathbb{B} = [B_1, \ldots, B_n] = [b_{ij}^{(k)}]_{i,j,k},$$

*respectively. There is a* $K$*-algebra isomorphism* $R_{\mathbb{A}} \cong R_{\mathbb{B}}$ *if and only if there exist a matrix* $T = (t_{ij}) \in \mathbb{M}_n(K)$ *and a permutation* $\sigma : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ *of the set* $\{1, \ldots, n\}$ *such that*

$$t_{ij} \neq 0, \quad t_{ii} = 1, \quad a_{\sigma(i)\sigma(j)}^{(\sigma(k))} t_{ij} = b_{ij}^{(k)} t_{ik} t_{kj}, \quad \text{for all } 1 \leq i, j, k \leq n.$$

*Proof.* Suppose that there is a $K$-algebra isomorphism $f : R_{\mathbb{A}} \rightarrow R_{\mathbb{B}}$. Then $f(u_{11}), \ldots, f(u_{nn})$ are orthogonal primitive idempotents of $R_{\mathbb{B}}$ such that $1_{R_{\mathbb{B}}} = f(u_{11}) + \cdots + f(u_{nn})$. It follows from [4, Theorem 3.4.1] that there exist a permutation $\sigma$ of the set $\{1, \ldots, n\}$ and an invertible element $b \in R_{\mathbb{B}}$ such that $v_{ii} = bf(u_{\sigma(i)\sigma(i)})b^{-1}$, for all $1 \leq i \leq n$. Hence there is a $K$-algebra isomorphism $g : R_{\mathbb{A}} \rightarrow R_{\mathbb{B}}$ such that $v_{ii} = g(u_{\sigma(i)\sigma(i)})$, for all $1 \leq i \leq n$. Since $g(u_{\sigma(i)\sigma(j)}) = v_{ii} g(u_{\sigma(i)\sigma(j)}) v_{jj}$, then $g(u_{\sigma(i)\sigma(j)}) = t_{ij} v_{ij}$, for some $0 \neq t_{ij} \in K$ $(1 \leq i, j \leq n)$. Clearly, $t_{ii} = 1$, for all $1 \leq i \leq n$. Since

$$g(u_{\sigma(i)\sigma(k)} u_{\sigma(k)\sigma(j)}) = g(u_{\sigma(i)\sigma(k)}) g(u_{\sigma(k)\sigma(j)}),$$

then we have $a_{\sigma(i)\sigma(j)}^{(\sigma(k))} t_{ij} = b_{ij}^{(k)} t_{ik} t_{kj}$, for all $1 \leq i, j, k \leq n$. It follows that $T := (t_{ij}) \in \mathbb{M}_n(K)$ is the desired matrix.

Conversely, suppose that there exist a matrix $T = (t_{ij})$ and a permutation $\sigma$ of the set $\{1, \ldots, n\}$ satisfying the above condition. Then the $K$-linear map

$$f : R_{\mathbb{A}} \rightarrow R_{\mathbb{B}},$$

given by $u_{ij} \mapsto t_{\sigma^{-1}(i)\sigma^{-1}(j)} v_{\sigma^{-1}(i)\sigma^{-1}(j)}$, defines a $K$-algebra isomorphism. $\square$

As an immedeate consequence of the proposition, we have the following.

**Corollary 2.2.** *Let* $R_{\mathbb{A}} = \bigoplus_{i,j=1}^{n} Ku_{ij}$ *and* $R_{\mathbb{B}} = \bigoplus_{i,j=1}^{n} Kv_{ij}$ *be full matrix algebras with* $(0,1)$*-structure systems*

$$\mathbb{A} = [A_1, \dots, A_n] = [a_{ij}^{(k)}]_{i,j,k} \quad \text{and} \quad \mathbb{B} = [B_1, \dots, B_n] = [b_{ij}^{(k)}]_{i,j,k},$$

*respectively. Then* $R_{\mathbb{A}}$ *is isomorphic to* $R_{\mathbb{B}}$ *as* $K$*-algebras if and only if there exists a permutation* $\sigma$ *of the set* $\{1, \dots, n\}$ *such that* $b_{ij}^{(k)} = a_{\sigma(i)\sigma(j)}^{(\sigma(k))}$ *for all* $1 \le i, j, k \le n$.

**Lemma 2.3.** *Let* $n \ge 3$ *be an integer, and let* $\mathbb{A} = [A_1, \dots, A_n] = [a_{ij}^{(k)}]_{i,j,k}$ *be a structure system. Then, for any distinct* $1 \le i, j, k \le n$*, the following equalities hold* $a_{ij}^{(k)} a_{ik}^{(j)} = 0$ *and* $a_{kj}^{(i)} a_{ij}^{(k)} = 0$.

*Proof.* This follows from (A1) and (A3). □

**Example 2.4.** By applying Lemma 2.3 and Corollary 2.2, one can verify that, for $n = 3$, the following five $(0,1)$-structure systems $\mathbb{A}^{(1)}$, $\mathbb{A}^{(2)}$, $\mathbb{A}^{(3)}$, $\mathbb{A}^{(4)}$, $\mathbb{A}^{(5)}$:

$$\left[\begin{array}{ccccccccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}\right], \left[\begin{array}{ccccccccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}\right],$$

$$\left[\begin{array}{ccccccccc} 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}\right], \left[\begin{array}{ccccccccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \end{array}\right],$$

$$\left[\begin{array}{ccccccccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}\right].$$

provide a list of all $(0,1)$-structure systems $\mathbb{A}$ such that every $\mathbb{A}$-full matrix algebra $R_{\mathbb{A}}$ is isomorphic to any of the algebras $R_{\mathbb{A}^{(1)}}$, $R_{\mathbb{A}^{(2)}}$, $R_{\mathbb{A}^{(3)}}$, $R_{\mathbb{A}^{(4)}}$, $R_{\mathbb{A}^{(5)}}$.

Given an arbitrary structure system $\mathbb{A} = [A_1, \dots, A_n] = [a_{ij}^{(k)}]_{i,j,k}$, we define a new one $\overline{\mathbb{A}} = [\overline{A}_1, \dots, \overline{A}_n] = [\overline{a}_{ij}^{(k)}]_{i,j,k}$, where

$$\overline{a}_{ij}^{(k)} := \begin{cases} 1, & \text{if } a_{ij}^{(k)} \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that $\overline{\mathbb{A}}$ is a structure system. Following [7, Definition 3.1], we call the $\overline{\mathbb{A}}$-full matrix algebra $R_{\overline{A}}$ a $(0,1)$-*limit* of $R_{\mathbb{A}}$.

**Theorem 2.5.** *For $n = 3$, there are just five $3 \times 3$ $\mathbb{A}$-full matrix algebras $R_{\mathbb{A}}$, up to isomorphism, which are given by the five $(0,1)$-structure systems in Example* 2.4.

*Proof.* Let $\mathbb{A}$ be a $3 \times 3$ $\mathbb{A}$-full matrix algebra, where $\mathbb{A} = [A_1, A_2, A_3] = [a_{ij}^{(k)}]$, and let $R_{\overline{A}}$ be the $(0,1)$-limit of $R_{\mathbb{A}}$. Then we show that $R_{\mathbb{A}}$ is isomorphic to $R_{\overline{\mathbb{A}}}$, using Proposition 2.1. We put $\sigma = \mathrm{id}$ and $T = (t_{ij}) \in \mathbb{M}_3(K)$, where

$$t_{ij} := \begin{cases} a_{ij}^{(k)}, & \text{if } a_{ij}^{(k)} \neq 0, \text{ for } k \neq i, j, \\ 1, & \text{otherwise}, \end{cases}$$

for distinct $i, j \in \{1, 2, 3\}$, and $t_{ii} := 1$ for $i = 1, 2, 3$. Then using Lemma 2.3, one can check that $\overline{a}_{ij}^{(k)} t_{ij} = a_{ij}^{(k)} t_{ik} t_{kj}$, for all $1 \leq i, j, k \leq n$. This completes the proof. □

## 3. Frobenius $\mathbb{A}$-full matrix algebras

In this section, we improve the characterization of Frobenius $\mathbb{A}$-full matrix algebras $R_{\mathbb{A}}$ given by [5, Lemma 4.2], where structure systems are $(0,1)$-matrices.

Assume that $R_{\mathbb{A}}$ is an $\mathbb{A}$-full matrix algebra (1.1) and let $M$ be a right $R_{\mathbb{A}}$-module with $\underline{\dim}\, M = (1, \ldots, 1)$. Then $M$ has a $K$-basis $\{v_1, \ldots, v_n\}$ such that $v_i u_{ii} = v_i$, for all $1 \leq i \leq n$. Consider the matrix $S = (s_{ij}) \in \mathbb{M}_n(K)$ such that

$$(*) \qquad v_i u_{kj} = \begin{cases} s_{ij} v_j, & \text{if } \ k = i, \\ 0, & \text{otherwise}, \end{cases}$$

for all $1 \leq i, j, k \leq n$, and that

$$(**) \quad s_{ii} = 1 \quad \text{and} \quad s_{ik} s_{kj} = a_{ij}^{(k)} s_{ij}, \quad \text{for all } \ 1 \leq i, j, k \leq n.$$

We call $S$ a *representation matrix* of $M$ with respect to a $K$-basis $\{v_1, \ldots, v_n\}$. Conversely, let $M$ be a $K$-vector space with a $K$-basis $\{v_1, \ldots, v_n\}$ and $S = (s_{ij}) \in \mathbb{M}_n(K)$ which satisfies the condition $(**)$. Then, by $(*)$, $M$ has a right $R_{\mathbb{A}}$-module structure with $\underline{\dim}\, M = (1, \ldots, 1)$, see [5, Proposition 2.1].

Now we modify [5, Propositions 2.2, 2.3 and Lemma 4.2] to remove the assumption of $(0,1)$-structure systems. We begin with the following lemma.

**Lemma 3.1.** *Assume that $R_{\mathbb{A}}$ is an $\mathbb{A}$-full matrix algebra* (1.1) *and let $M$, $M'$ be right $R_{\mathbb{A}}$-modules, with $\underline{\dim}\, M = \underline{\dim}\, M' = (1, \ldots, 1)$ and with* *the representation matrices $S = (s_{ij})$ and $S' = (s'_{ij})$, respectively. There exists an isomorphism $M \cong M'$ of right $R_{\mathbb{A}}$-modules if and only if there exist $t_1, \ldots, t_n \in K$ such that*

$$t_i \neq 0 \quad \text{and} \quad s_{ij}t_j = t_i s'_{ij}, \quad \text{for all } i, j \in \{1, \ldots, n\}.$$

*Proof.* Let $\{v_i \mid 1 \leq i \leq n\}, \{v'_i \mid 1 \leq i \leq n\}$ be associated $K$-bases of $M, M'$ with representation matrices $S = (s_{ij})$ and $S' = (s'_{ij})$, respectively.

First suppose that there is an isomorphism $f : M \to M'$. Since $v'_j = v'_j u_{jj}$, for all $j \in \{1, \ldots, n\}$, then there exists $0 \neq t_i \in K$ such that $f(v_i) = f(v_i)u_{ii} = t_i v'_i$, for each $i \in \{1, \ldots, n\}$. The equality $f(v_i u_{ij}) = f(v_i)u_{ij}$ yields $s_{ij}t_j = t_i s'_{ij}$, for all $i, j \in \{1, \ldots, n\}$.

Conversely, suppose that there exist $t_1, \ldots, t_n \in K$ satisfying the above conditions. Since $t_i \neq 0$, for all $i \in \{1, \ldots, n\}$, we can define a $K$-linear isomorphism $f : M \to M'$ by $f(v_i) := t_i v'_i$, for all $i \in \{1, \ldots, n\}$. The latter condition implies that $f$ is an $R_{\mathbb{A}}$-module homomorphism, so that $f : M \to M'$ is an isomorphism. □

Indecomposable projective $R_{\mathbb{A}}$-modules are characterized by their representation matrices as follows, see [5, Proposition 2.2].

**Lemma 3.2.** *Assume that $R_{\mathbb{A}}$ is an $\mathbb{A}$-full matrix algebra* (1.1).

(i) *For each indecomposable projective right $R_{\mathbb{A}}$-module $u_{ii}R_{\mathbb{A}}$, we have*

- $\underline{\dim}\, u_{ii}R_{\mathbb{A}} = (1, \ldots, 1)$ *and*
- *the representation matrix of the module $u_{ii}R_{\mathbb{A}}$, with respect to the $K$-basis $\{u_{ij} \mid 1 \leq j \leq n\}$, is the $n \times n$ matrix $(a^{(k)}_{ij})_{k,j}$, where the $(k, j)$-entry equals $a^{(k)}_{ij}$.*

(ii) *Let $M$ be a right $R_{\mathbb{A}}$-module with $\underline{\dim} M = (1, \ldots, 1)$, and let $S = (s_{ij})$ be a representation matrix of $M$ with respect to a $K$-basis $\{v_i \mid 1 \leq i \leq n\}$. Then $M$ is isomorphic to to the projective $R_{\mathbb{A}}$-module $u_{ll}R_{\mathbb{A}}$ if and only if $s_{lk} \neq 0$, for all $k \in \{1, \ldots, n\}$.*

*Proof.* (i) This follows from the definition of the multiplication of $R_{\mathbb{A}}$, that is, $u_{ik}u_{kj} = a^{(k)}_{ij} u_{ij}$, for all $i, j, k \in \{1, \ldots, n\}$. Note that (A2) implies $\underline{\dim}\, u_{ii}R_{\mathbb{A}} = (1, \ldots, 1)$.

(ii) First suppose that $M$ is isomorphic to $u_{ll}R_{\mathbb{A}}$. Then it follows from Lemma 3.1 that there exist $t_1, \ldots, t_n \in K$ such that $t_i \neq 0$ and $s_{ij}t_j = t_i a^{(i)}_{lj}$, for all $i, j \in \{1, \ldots, n\}$. Hence $s_{lj}t_j = t_i a^{(l)}_{lj} = t_i \neq 0$, so that $s_{lj} \neq 0$, for all $j \in \{1, \ldots, n\}$.

Conversely, suppose that $s_{lj} \neq 0$, for all $j \in \{1, \ldots, n\}$. Since $a_{lj}^{(i)} s_{lj} = s_{li} s_{ij}$, for all $i, j \in \{1, \ldots, n\}$, then there is an $R_{\mathbb{A}}$-module isomorphism $f : u_{ll} R_{\mathbb{A}} \to M,\ u_{lj} \mapsto s_{lj} v_j\ (1 \leq j \leq n)$. □

We denote the standard duality functor $\mathrm{Hom}_K(-, K) : \mathrm{mod}\, R_{\mathbb{A}} \to \mathrm{mod}\, R_{\mathbb{A}}^{op}$ by $(-)^*$. As a dual of Lemma 3.2, we obtain the following, see [5, Proposition 2.3].

**Lemma 3.3.** *Assume that $R_{\mathbb{A}}$ is an $\mathbb{A}$-full matrix algebra* (1.1).

(i) *For each indecomposable injective right $R_{\mathbb{A}}$-module $(R_{\mathbb{A}} u_{jj})^*$, we have*

- $\underline{\dim}\,(R_{\mathbb{A}} u_{jj})^* = (1, \ldots, 1)$ *and*
- *the representation matrix of the module $(R_{\mathbb{A}} u_{jj})^*$, with respect to the dual $K$-basis $\{u_{ij}^* \mid 1 \leq i \leq n\}$, is the $n \times n$ matrix $(a_{ij}^{(k)})_{i,k}$, where the $(i, k)$-entry equals $a_{ij}^{(k)}$.*

(ii) *Let $M$ be a right $R_{\mathbb{A}}$-module with $\underline{\dim}\, M = (1, \ldots, 1)$, and let $S = (s_{ij})$ be a representation matrix of $M$ with respect to a $K$-basis $\{v_i \mid 1 \leq i \leq n\}$. Then $M$ is isomorphic to the injective $R_{\mathbb{A}}$-module $(R_{\mathbb{A}} u_{ll})^*$ if and only if $s_{kl} \neq 0$, for all $k \in \{1, \ldots, n\}$.*

**Proposition 3.4.** *Let $R_{\mathbb{A}}$ be an $\mathbb{A}$-full $n \times n$ matrix algebra, where $\mathbb{A} = [A_1, \ldots, A_n]$ is the structure system and $A_k = (a_{ij}^{(k)})$ $(1 \leq k \leq n)$. The following two conditions are equivalent.*

(i) *$R_{\mathbb{A}}$ is a Frobenius algebra with Nakayama permutation $\sigma$.*

(ii) *There exists a permutation $\sigma$ of the set $\{1, \ldots, n\}$ such that $\sigma(i) \neq i$, for all $i \in \{1, \ldots, n\}$, and that $a_{i\sigma(i)}^{(k)} \neq 0$, for all $i, k \in \{1, \ldots, n\}$.*

*Proof.* (i)⇒(ii) It follows from (i) that $u_{ii} R_{\mathbb{A}} \cong (R_{\mathbb{A}} u_{\sigma(i)\sigma(i)})^*$, for all $i \in \{1, \ldots, n\}$. Since $u_{ii} R_{\mathbb{A}}$ has a representation matrix $(a_{ij}^{(k)})_{k,j}$ with respect to a $K$-basis $\{u_{i1}, \ldots, u_{in}\}$ then Lemma 3.3 yields $a_{i\sigma(i)}^{(k)} \neq 0$, for all $i, k \in \{1, \ldots, n\}$. Since $\underline{\dim}\, u_{ii} R_{\mathbb{A}} = (1, \ldots, 1)$ then $\mathrm{soc}(u_{ii} R_{\mathbb{A}}) \not\cong u_{ii} R_{\mathbb{A}} / u_{ii} J(R_{\mathbb{A}})$, so that $\sigma(i) \neq i$, for all $i \in \{1, \ldots, n\}$.

(ii)⇒(i) Lemmas 3.2 and 3.3 yield the isomorphism $u_{ii} R_{\mathbb{A}} \cong (R_{\mathbb{A}} u_{\sigma(i)\sigma(i)})^*$ of right $R_{\mathbb{A}}$-modules, for all $1 \leq i \leq n$. Hence (i) follows. □

## 4. Infinite families of $\mathbb{A}$-full matrix algebras

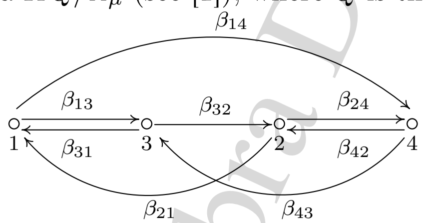
In this section, for $n = 4, 5$ and $n = 6$, we construct several interesting infinite families of $\mathbb{A}$-full matrix algebras $R_{\mathbb{A}}$ that are of infinite representation type. We also determine their representation type (tame or wild), by applying the well-known representation theory diagrammatic criteria, see [1], [11] and [12]. We end the section by presenting an idea of a construction of a large class of Frobenius $\mathbb{A}$-full matrix algebras $R_{\mathbb{A}}$ such that $\dim_K R_{\mathbb{A}} = n^2$. $n \geq 4$, $\operatorname{soc} R_{\mathbb{A}} = J(R_{\mathbb{A}})^{n-2}$ and $J(R_{\mathbb{A}})^{n-1} = 0$. A characterization of all Frobenius algebras $R_{\mathbb{A}}$ with the above properties remains an open problem.

**Example 4.1.** Assume that $n = 4$ and $K$ is a field. Consider the one-parameter family of $\mathbb{A}_\mu$-full matrix algebras $C_\mu = R_{\mathbb{A}_\mu}$, where $\mu \in K^* = K \setminus \{0\}$ and $\mathbb{A}_\mu$ is the following structure system

$$\mathbb{A}_\mu = \begin{bmatrix} 1111 & 0100 & 0110 & 0101 \\ 10\mu0 & 1111 & 0010 & 0011 \\ 1001 & 0101 & 1111 & 0001 \\ 1000 & 1100 & 1010 & 1111 \end{bmatrix}.$$

A simple calculation shows that, given $\mu \in K^*$, the matrix satisfies the conditions (A1)–(A3). We show that the algebra $C_\mu$ is isomorphic to the bound quiver $K$-algebra $KQ/\Omega_\mu$ (see [1]), where $Q$ is the quiver

$Q$ :




and $\Omega_\mu$ is the two-sided ideal of the path $K$-algebra $KQ$ of $Q$ generated by the following relations:

- $\beta_{21}\beta_{13} - \mu \cdot \beta_{24}\beta_{43}$,
- $\beta_{13}\beta_{32} - \beta_{14}\beta_{42}$,
- $\beta_{32}\beta_{24} - \beta_{31}\beta_{14}$,
- $\beta_{43}\beta_{31} - \beta_{42}\beta_{21}$,
- $\beta_{13}\beta_{31}$, $\beta_{31}\beta_{13}$, $\beta_{24}\beta_{42}$, $\beta_{42}\beta_{24}$,
- $\beta_{21}\beta_{14}$, $\beta_{43}\beta_{32}$, $\beta_{32}\beta_{21}$, $\beta_{14}\beta_{43}$.

It is easy to check that the correspondences $\varepsilon_j \mapsto u_{jj}$ and $\beta_{ij} \mapsto u_{ij}$ define a $K$-algebra homomorphism $h : KQ/\Omega_\mu \to C_\mu$, where $\varepsilon_j$ is the primitive idempotent of the path algebra $KQ$ defined by the stationary path at the vertex $j$, for every $j \in (Q_\mu)_0$. Note that $\dim_K KQ/\Omega_\mu = 16$ and the cosets of the idempotents $\overline{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4}$, the eigth arrows $\overline{\beta_{ij}} \in (Q_\mu)_1$, together with the four cosets $\overline{\beta_{21}\beta_{13}}$, $\overline{\beta_{13}\beta_{32}}$, $\overline{\beta_{32}\beta_{24}}$, and $\overline{\beta_{43}\beta_{31}}$ form a $K$-basis of the quotient $K$-algebra $KQ/\Omega_\mu$. Since

$e_{23} = h(\overline{\beta_{21}\beta_{13}})$, $e_{12} = h(\overline{\beta_{13}\beta_{32}})$, $e_{34} = h(\overline{\beta_{32}\beta_{24}})$, and $e_{41} = h(\overline{\beta_{43}\beta_{31}})$

then the map $h$ is surjective. Finally, since $\dim_K KQ/\Omega_\mu = \dim_K C_\mu = 16$, the surjection is an isomorphism of $K$-algebras.

It follows from the shape of $Q$ and $\Omega_\mu$ that, for each $\mu \in K^*$, $KQ/\Omega_\mu \cong C_\mu$ is a special biserial algebra [13], and therefore it is representation-tame, see [3, 5.2]. Since there is a cyclic walk

$$1 \xrightarrow{\beta_{13}} 3 \xleftarrow{\beta_{43}} 4 \xrightarrow{\beta_{42}} 2 \xleftarrow{\beta_{32}} 3 \xrightarrow{\beta_{31}} 1 \xleftarrow{\beta_{21}} 2 \xrightarrow{\beta_{24}} 4 \xleftarrow{\beta_{14}} 1$$

of the quiver $Q$ then, according to the finite representation type criterion in [13] (see see also [10, Proposition 3.7]), the algebra $C_\mu$ is of infinite representation type. Note also that, for each $\mu \in K^*$, $C_\mu$ is self injective, $J(C_\mu)^3 = 0$ and

$$J(C_\mu)^2 = \mathrm{soc}(C_\mu) = K\overline{\beta_{21}\beta_{13}} \oplus K\overline{\beta_{13}\beta_{32}} \oplus K\overline{\beta_{32}\beta_{24}} \oplus K\overline{\beta_{43}\beta_{31}},$$

see also [7, Section 5]. Consequently, the quotient algebras

$$\overline{C}_\mu = C_\mu/\mathrm{soc}\, C_\mu \quad \text{and} \quad \overline{C}_\gamma = C_\gamma/\mathrm{soc}\, C_\gamma$$

are isomorphic, for each pair $\mu, \gamma \in K^*$. In particular, it follows that the numbers of the indecomposable $\overline{C}_\mu$-modules and $\overline{C}_\gamma$-modules are equal and the stable Auslander-Reiten quivers of $\overline{C}_\mu$ and of $\overline{C}_\gamma$ are isomorphic.

**Example 4.2.** Assume that $n = 6$. Consider the one-parameter family of $\mathbb{A}_\mu$-full matrix algebras $H_\mu = R_{\mathbb{A}_\mu}$, where $\mu \in K$ and

$$\mathbb{A}_\mu = \begin{bmatrix} 111111 & 010000 & 011000 & 010100 & 011110 & 011101 \\ 100000 & 111111 & 001000 & 000100 & 001110 & 001101 \\ 100111 & 010111 & 111111 & 000100 & 000110 & 000101 \\ 101011 & 011011 & 001000 & 111111 & 001010 & 001001 \\ 100000 & 010000 & \mu 11000 & 110100 & 111111 & 000001 \\ 100010 & 010010 & 111010 & 110110 & 000010 & 111111 \end{bmatrix}.$$

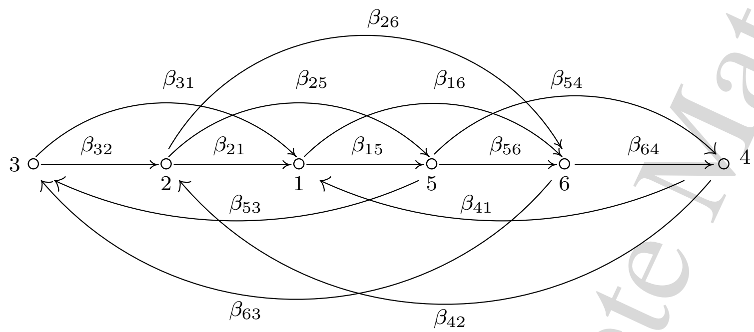
First we observe that:

(a) if $K$ is infinite, then the family $\{H_\mu\}_{\mu \in K\setminus\{0,1\}}$ is infinite, because $H_\mu \cong H_\gamma$ if and only $\mu = \gamma$, for $\mu, \gamma \in K \setminus \{0,1\}$ (apply Corollary 2.2),

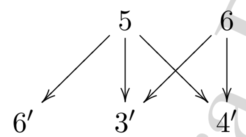
(b) for each $\mu \in K \setminus \{0,1\}$, the algebra $H_\mu$ is not self-injective (the right ideals $u_{22}H_\mu$ and $u_{55}H_\mu$ are not injective, by Lemma 3.3, and

(c) for each $\mu \in K \setminus \{0,1\}$, the Gabriel quiver $\mathcal{Q}(H_\mu)$ of the algebra $H_\mu$ is the following quiver $Q$ (apply [5, Proposition 1.2]).

$Q$ :




Now we show that, for each $\mu \in K^*$, the algebra $H_\mu$ is of wild representation type, see [9, Section 14.2] and [12, Chapter XIX]. To see this, we note that $H_\mu/J(H_\mu)^2 \cong H_\nu/J(H_\nu)^2$, for all $\mu, \nu \in K^*$, and the algebra $B := H_\mu/J(H_\mu)^2$ has $J(B)^2 = 0$. It follows that $\mathcal{Q}(B) = \mathcal{Q}(H_{q_\mu})$. Since the separated quiver $\mathcal{Q}^s(B)$ of $B$ (see [2, Section X.2]) contains a wild subquiver of the form




then, by [9, Theorems 14.14 and 14.15] and [12, Chapter XIX], the algebra $B$ is representation-wild and hence also $H_\mu$ is representation-wild, for each $\mu \in K^*$, because there is a fully faithful exact embedding $\text{mod } B \hookrightarrow \text{mod } H_\mu$.

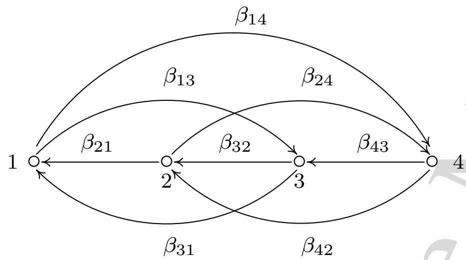
**Example 4.3.** Assume that $n = 4$, $K$ is a field and $\mathbb{A}$ is a structure system such that $R_\mathbb{A}$ is a Frobenius algebra and the Nakayama permutation of $R_\mathbb{A}$ is the cyclic permutation $\sigma = (1, 2, 3, 4)$, see [5, Theorem 3.4] and [7, Theorem 5.5]. The structure system $\mathbb{A}$ and the associated $(0, 1)$-structure system $\overline{\mathbb{A}}$ have the following forms

$$\mathbb{A} = \left[\begin{array}{cccc|cccc|cccc|cccc} 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & \mu_6 & 1 & 0 & 0 & \mu_7 & 0 & 1 \\ 1 & 0 & \mu_1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & \mu_8 & 1 \\ 1 & 0 & 0 & \mu_2 & 0 & 1 & 0 & \mu_4 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & \mu_3 & 1 & 0 & 0 & \mu_5 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \end{array}\right].$$

$$\overline{\mathbb{A}} = \left[\begin{array}{cccc|cccc|cccc|cccc} 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \end{array}\right],$$

where $\mu_1, \ldots, \mu_8$ are arbitrary scalars in $K \setminus \{0\}$. By Proposition 3.4 (see also [5, Theorem 3.4] and [7, Theorem 5.5]), each of the algebra $R_\mathbb{A}$ in the defined eight parameter family is Frobenius and $\operatorname{soc} R_\mathbb{A} = J(R_\mathbb{A})^2$. One shows that there is a $K$-algebra isomorphism $R_{\overline{\mathbb{A}}} \cong KQ/\Omega_\sigma$, where $Q$ is the quiver

$Q$ :




and $\Omega_\sigma$ is the two-sided ideal of the path $K$-algebra $KQ$ of $Q$ generated by the following elements:

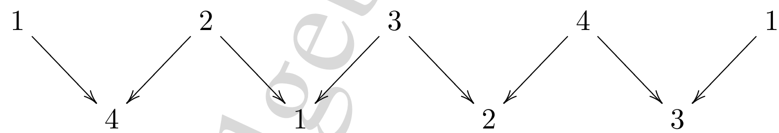
1° $\beta_{21}\beta_{13}-\beta_{24}\beta_{43}$, $\beta_{13}\beta_{32}-\beta_{14}\beta_{42}$, $\beta_{32}\beta_{24}-\beta_{31}\beta_{14}$, $\beta_{43}\beta_{31}-\beta_{42}\beta_{21}$,

2° $\beta_{13}\beta_{31}$, $\beta_{31}\beta_{13}$, $\beta_{24}\beta_{42}$, $\beta_{42}\beta_{24}$, $\beta_{21}\beta_{14}$, $\beta_{14}\beta_{43}$, $\beta_{32}\beta_{21}$, and $\beta_{43}\beta_{32}$.

It follows that

- the zero relation $\alpha_1\alpha_2\alpha_3$ belongs to $\Omega_\sigma$, for each path $\bullet \xrightarrow{\alpha_1} \bullet \xrightarrow{\alpha_2} \bullet \xrightarrow{\alpha_3} \bullet$ in $Q$,
- there is an algebra isomorphism $\overline{R}_{\mathbb{A}} \cong KQ/\Omega_\sigma$, and
- $J(\overline{R}_{\mathbb{A}})^3 = 0$ and $J(\overline{R}_{\mathbb{A}})^2 = \operatorname{soc}(\overline{R}_{\mathbb{A}})$.

Now it is easy to see that $\overline{R}_{\mathbb{A}}$ is a special biserial algebra, and therefore it is representation-tame, see [3, 5.2]. Since there is a cyclic walk




in $Q$ then, according to [13], the Frobenius algebra $\overline{R}_{\mathbb{A}}$ is of infinite representation type, see also [10, Proposition 3.7].

We end this section by presenting an idea of a construction, for $n = 5$, of tame Frobenius $\mathbb{A}$-full matrix algebras $R_{\mathbb{A}}$ of infinite representation type such that $J(R_{\mathbb{A}})^4 = 0$ and $J(R_{\mathbb{A}})^3 = \operatorname{soc} R_{\mathbb{A}}$.

**Example 4.4.** Assume that $n = 5$ and $K$ is a field. We construct a set of structure systems $q = [q^{(1)}, q^{(2)}, q^{(3)}, q^{(4)}, q^{(5)}]$ such that $R_q$ is a Frobenius algebra, $J(R_q)^4 = 0$, $J(R_q)^3 = \operatorname{soc} R_q)$, and $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}$ is the Nakayama permutation of $R_q$.

Suppose that $q = [q^{(1)}, q^{(2)}, q^{(3)}, q^{(4)}, q^{(5)}] = [q_{ij}^{(k)}]_{i,j,k}$ is such a structure system and let

$$R_q = \bigoplus_{i,j=1}^{5} Ke_{ij}$$

be the corresponding $q$-full matrix $K$-algebra with the basis $\{e_{ij} \mid 1 \leq i, j \leq 5\}$. We recall that the elements $e_1 = e_{11}, \ldots, e_5 = e_{55}$ form a complete set of pairwise orthogonal primitive idempotents of the algebra $R_q$ and $1 = e_1 + e_2 + e_3 + e_4 + e_5$ is the identity of $R_q$. We denote by $\cdot_q$ the multiplication in $R_q$.

One shows that $\mathrm{soc}(e_j R_q) = K e_{j\,\sigma(j)}$ (see [7, Theorem 5.3]) and therefore $e_{j\,\sigma(j)} \cdot_q J(R_q) = 0$ and $J(R_q) \cdot_q e_{j\,\sigma(j)} = 0$, for $j = 1, \ldots, 5$. Hence we get the equalities $q_{jr}^{(\sigma(j))} = 0$, for all $r \neq \sigma(j)$, and $q_{s\,\sigma(j)}^{(j)} = 0$, for all $s \neq j$, that is,

- $q_{32}^{(1)} = q_{34}^{(1)} = q_{35}^{(1)} = 0$ and $q_{34}^{(1)} = q_{24}^{(1)} = q_{54}^{(1)} = 0$,
- $q_{41}^{(2)} = q_{43}^{(2)} = q_{45}^{(2)} = 0$ and $q_{45}^{(2)} = q_{35}^{(2)} = q_{15}^{(2)} = 0$,
- $q_{51}^{(3)} = q_{52}^{(3)} = q_{54}^{(3)} = 0$ and $q_{51}^{(3)} = q_{41}^{(3)} = q_{21}^{(3)} = 0$,
- $q_{12}^{(4)} = q_{13}^{(4)} = q_{15}^{(4)} = 0$ and $q_{12}^{(4)} = q_{52}^{(4)} = q_{32}^{(4)} = 0$,
- $q_{23}^{(5)} = q_{24}^{(5)} = q_{21}^{(5)} = 0$ and $q_{23}^{(5)} = q_{43}^{(5)} = q_{13}^{(5)} = 0$.

Consequently, the block matrix $q$ has the form

$$q = \left[\begin{array}{ccccc|ccccc|ccccc|ccccc|ccccc} 1&1&1&1&1&0&1&*&*&0&0&*&1&*&*&0&0&0&1&0&0&*&0&*&1 \\ 1&0&*&0&*&1&1&1&1&1&0&0&1&*&*&*&0&*&1&*&0&0&0&0&1 \\ 1&0&0&0&0&*&1&0&*&0&1&1&1&1&1&*&0&0&1&*&*&*&0&*&1 \\ 1&*&*&0&*&0&1&0&0&0&0&*&1&0&*&1&1&1&1&1&*&*&0&0&1 \\ 1&*&*&0&0&*&1&*&*&0&0&0&1&0&0&*&0&*&1&0&1&1&1&1&1 \end{array}\right].$$

Since we assume that $\mathrm{soc}(e_j R_q) = K e_{j\,\sigma(j)} \subseteq e_j J(R_q)^3$, for $j = 1, \ldots, 5$, then $K e_{j\,\sigma(j)} = K(e_{j\,j_1} \cdot_q e_{j_1 j_2} \cdot_q e_{j_2\,\sigma(j)})$, where $j_1 \neq j_2$ and $j_1, j_2 \notin \{j, \sigma(j)\}$.

Assume, for simplicity, that there exist non-zero scalars $\lambda_{14}, \lambda_{25}, \lambda_{31}, \lambda_{42}, \lambda_{53} \in K$ such that

$$\lambda_{14} e_{14} = e_{12} \cdot_q e_{23} \cdot_q e_{34},$$
$$\lambda_{25} e_{25} = e_{23} \cdot_q e_{34} \cdot_q e_{45},$$
$$\lambda_{31} e_{31} = e_{34} \cdot_q e_{45} \cdot_q e_{51},$$
$$\lambda_{42} e_{42} = e_{45} \cdot_q e_{51} \cdot_q e_{12},$$
$$\lambda_{53} e_{53} = e_{51} \cdot_q e_{12} \cdot_q e_{23}.$$

Hence we conclude that

$q_{12}^{(3)} = q_{12}^{(5)} = 0$, $q_{34}^{(2)} = q_{34}^{(5)} = 0$, $q_{23}^{(1)} = q_{23}^{(4)} = 0$, $q_{45}^{(1)} = q_{45}^{(3)} = 0$ , $q_{51}^{(2)} = q_{51}^{(4)} = 0$.

Indeed, if we assume to the contrary that $q_{12}^{(3)} \neq 0$ then $e_{13} \cdot_q e_{32} = q_{12}^{(3)} e_{12}$ and then the non-zero element $\lambda_{14} q_{12}^{(3)} e_{14} = e_{13} \cdot_q e_{32} \cdot_q e_{23} \cdot_q e_{34}$ belongs to $J(R_q)^4 = 0$, and we get a contradiction. The remaining equalities follow in a similar way.

Moreover, since the elements $\lambda_{14}, \lambda_{25}, \lambda_{31}, \lambda_{42}, \lambda_{53} \in K$ are non-zero then, by the associativity of $\cdot_q$, the equalities above yields

$$q_{13}^{(2)} q_{14}^{(3)} = q_{14}^{(2)} q_{24}^{(3)} \neq 0, \quad q_{24}^{(3)} q_{25}^{(4)} = q_{25}^{(3)} q_{35}^{(4)} \neq 0, \quad q_{35}^{(4)} q_{31}^{(5)} = q_{31}^{(4)} q_{41}^{(5)} \neq 0,$$
$$q_{41}^{(5)} q_{42}^{(1)} = q_{42}^{(5)} q_{52}^{(1)} \neq 0, \quad q_{52}^{(1)} q_{53}^{(2)} = q_{53}^{(1)} q_{13}^{(2)} \neq 0.$$

Equivalently, we get the equalities

$$
\begin{aligned}
q_{24}^{(3)} &= \frac{q_{13}^{(2)} q_{14}^{(3)}}{q_{14}^{(2)}},\\
q_{35}^{(4)} &= \frac{q_{24}^{(3)} q_{25}^{(4)}}{q_{25}^{(3)}} = \frac{q_{13}^{(2)} q_{14}^{(3)} q_{25}^{(4)}}{q_{14}^{(2)} q_{25}^{(3)}},\\
q_{41}^{(5)} &= \frac{q_{35}^{(4)} q_{31}^{(5)}}{q_{31}^{(4)}} = \frac{q_{13}^{(2)} q_{14}^{(3)} q_{25}^{(4)} q_{31}^{(5)}}{q_{14}^{(2)} q_{25}^{(3)} q_{31}^{(4)}},\\
q_{52}^{(1)} &= \frac{q_{41}^{(5)} q_{42}^{(1)}}{q_{42}^{(5)}} = \frac{q_{13}^{(2)} q_{14}^{(3)} q_{25}^{(4)} q_{31}^{(5)} q_{42}^{(1)}}{q_{14}^{(2)} q_{25}^{(3)} q_{31}^{(4)} q_{42}^{(5)}},\\
q_{13}^{(2)} &= \frac{q_{52}^{(1)} q_{53}^{(2)}}{q_{53}^{(1)}} = \frac{q_{13}^{(2)} q_{14}^{(3)} q_{25}^{(4)} q_{31}^{(5)} q_{42}^{(1)} q_{53}^{(2)}}{q_{14}^{(2)} q_{25}^{(3)} q_{31}^{(4)} q_{42}^{(5)} q_{53}^{(1)}}.
\end{aligned}
$$

It follows that if $q_{13}^{(2)} \in K^*$ is arbitrary, then the remaining non-zero scalars $q_{ij}^{(s)}$ that appear in the equalities above satisfy the condition

$$
(*) \qquad q_{14}^{(3)} q_{25}^{(4)} q_{31}^{(5)} q_{42}^{(1)} q_{53}^{(2)} = q_{14}^{(2)} q_{25}^{(3)} q_{31}^{(4)} q_{42}^{(5)} q_{53}^{(1)}
$$

Now we show that $q_{43}^{(1)} = 0$, $q_{54}^{(2)} = 0$, $q_{15}^{(3)} = 0$, $q_{21}^{(4)} = 0$ and $q_{32}^{(5)} = 0$. To see this, assume to the contrary that $q_{43}^{(1)} \neq 0$. Then $0 \neq q_{43}^{(1)} e_{43} = e_{41} \cdot_q e_{13}$. It follows that the non-zero element $e_{45} \cdot_q e_{51} \cdot_q e_{12} \cdot_q e_{23} = q_{41}^{(5)} q_{13}^{(2)} e_{41} \cdot_q e_{13} = q_{41}^{(5)} q_{13}^{(2)} q_{43}^{(1)} e_{43}$ belongs to $J(R_q)^4 = 0$, and we get a contradiction. The equalities $q_{54}^{(2)} = 0$, $q_{15}^{(3)} = 0$, $q_{21}^{(4)} = 0$, $q_{32}^{(5)} = 0$ follow in a similar way. Consequently, the block matrix $q$ has the form

$$
q = \left[\begin{array}{ccccc|ccccc|ccccc|ccccc|ccccc}
1 & 1 & 1 & 1 & 1 & 0 & 1 & q_{13}^{(2)} & q_{14}^{(2)} & 0 & 0 & 0 & 1 & q_{14}^{(3)} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & * & 1\\
1 & 0 & 0 & 0 & * & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & q_{24}^{(3)} & q_{25}^{(3)} & 0 & 0 & 0 & 1 & q_{25}^{(4)} & 0 & 0 & 0 & 0 & 1\\
1 & 0 & 0 & 0 & 0 & & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & q_{31}^{(4)} & 0 & 0 & 1 & q_{35}^{(4)} & q_{31}^{(5)} & 0 & 0 & 0 & 1\\
1 & q_{42}^{(1)} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & * & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & q_{41}^{(5)} & q_{42}^{(5)} & 0 & 0 & 1\\
1 & q_{52}^{(1)} & q_{53}^{(1)} & 0 & 0 & 0 & 1 & q_{53}^{(2)} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & * & 1 & 0 & 1 & 1 & 1 & 1 & 1
\end{array}\right].
$$

Now we claim that each of the scalars $q_{25}^{(1)}, q_{31}^{(2)}, q_{42}^{(3)}, q_{53}^{(4)}, q_{14}^{(5)}$ is non-zero. Assume, to the contrary, that some of them is zero, say $q_{14}^{(5)} = 0$. It follows from the shape of $q$ that $q_{1r}^{(5)} = 0$, for all $r \neq 5$, and consequently $e_{15} \cdot_q J(R_q) = 0$. It follows that $S' = e_{15}K \subseteq e_1 R_q$ is a simple submodule of $e_1 R_q$; contrary to the assumption that $\mathrm{soc}(e_1 R_q) = e_{14}K$. This finishes the proof of our claim. Consequently, the block matrix $q$ has the form

$$
q = \left[\begin{array}{ccccc|ccccc|ccccc|ccccc|ccccc}
1 & 1 & 1 & 1 & 1 & 0 & 1 & q_{13}^{(2)} & q_{14}^{(2)} & 0 & 0 & 0 & 1 & q_{14}^{(3)} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & q_{14}^{(5)} & 1\\
1 & 0 & 0 & 0 & q_{25}^{(1)} & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & q_{24}^{(3)} & q_{25}^{(3)} & 0 & 0 & 0 & 1 & q_{25}^{(4)} & 0 & 0 & 0 & 0 & 1\\
1 & 0 & 0 & 0 & 0 & q_{31}^{(2)} & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & q_{31}^{(4)} & 0 & 0 & 1 & q_{35}^{(4)} & q_{31}^{(5)} & 0 & 0 & 0 & 1\\
1 & q_{42}^{(1)} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & q_{42}^{(3)} & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & q_{41}^{(5)} & q_{42}^{(5)} & 0 & 0 & 1\\
1 & q_{52}^{(1)} & q_{53}^{(1)} & 0 & 0 & 0 & 1 & q_{53}^{(2)} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & q_{53}^{(4)} & 1 & 0 & 1 & 1 & 1 & 1 & 1
\end{array}\right].
$$

where $q_{25}^{(1)}, q_{31}^{(2)}, q_{42}^{(3)}, q_{53}^{(4)}, q_{14}^{(5)}$ and $q_{13}^{(2)}$ are arbitrary non-zero scalars in $K$, the coefficients

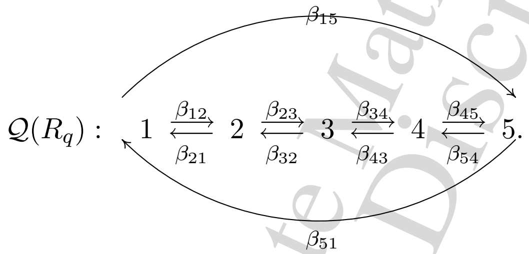
$$
q_{14}^{(3)}, q_{25}^{(4)}, q_{31}^{(5)}, q_{42}^{(1)}, q_{53}^{(2)}, q_{14}^{(2)}, q_{25}^{(3)}, q_{31}^{(4)}, q_{42}^{(5)}, q_{53}^{(1)}
$$

satisfy the equation $(*)$ and the coefficients $q_{52}^{(1)}, q_{13}^{(2)}, q_{24}^{(3)}, q_{35}^{(4)}, q_{41}^{(5)}$ depend of the remaining ones by the formulas preceding the equation $(*)$.

Conversely, if $q$ is a block matrix of the above form, where $q_{25}^{(1)}$, $q_{31}^{(2)}$, $q_{42}^{(3)}$, $q_{53}^{(4)}$, $q_{14}^{(5)}$ and $q_{13}^{(2)}$ are arbitrary non-zero scalars in $K$, and the remaining ones satisfy the above conditions then $q$ is a structure system and $R_q$ is a Frobenius algebra such that $J(R_q)^4 = 0$ and $\mathrm{soc}(R_q) = J(R_q)^3$. The associated $(0,1)$-matrix $\overline{q}$ structure system has the following form

$$\overline{q} = \left[\begin{array}{ccccc|ccccc|ccccc|ccccc|ccccc} 1&1&1&1&1&0&1&1&1&0&0&0&1&1&0&0&0&0&1&0&0&0&0&1&1\\ 1&0&0&0&1&1&1&1&1&1&0&0&1&1&1&0&0&0&1&1&0&0&0&0&1\\ 1&0&0&0&0&1&1&0&0&0&1&1&1&1&1&1&0&0&1&1&1&0&0&0&1\\ 1&1&0&0&0&0&1&0&0&0&0&1&1&0&0&1&1&1&1&1&1&1&0&0&1\\ 1&1&1&0&0&0&1&1&0&0&0&0&1&0&0&0&0&1&1&0&1&1&1&1&1 \end{array}\right].$$

It follows that the Gabriel quiver $\mathcal{Q}(R_q)$ of $R_q$ has the form




To view the algebra $R_q$ as a path algebra $KQ/\Omega_q$ of a bound quiver, we note that

$$\begin{aligned}
&q_{14}^{(5)} e_{12} \cdot_q e_{23} \cdot_q e_{34} = q_{13}^{(2)} q_{14}^{(3)} e_{15} \cdot_q e_{54},\\
&q_{25}^{(1)} e_{23} \cdot_q e_{34} \cdot_q e_{45} = q_{24}^{(3)} q_{25}^{(4)} e_{21} \cdot_q e_{15},\\
&q_{31}^{(2)} e_{34} \cdot_q e_{45} \cdot_q e_{51} = q_{35}^{(4)} q_{31}^{(5)} e_{32} \cdot_q e_{21},\\
&q_{42}^{(3)} e_{45} \cdot_q e_{51} \cdot_q e_{12} = q_{41}^{(5)} q_{42}^{(1)} e_{43} \cdot_q e_{32},\\
&q_{53}^{(4)} e_{51} \cdot_q e_{12} \cdot_q e_{23} = q_{52}^{(1)} q_{53}^{(2)} e_{54} \cdot_q e_{43}.
\end{aligned}$$

To see the first equality, we note that $e_{15} \cdot_q e_{54} = q_{14}^{(5)} e_{14}$ and $e_{12} \cdot_q e_{23} \cdot_q e_{34} = q_{13}^{(2)} q_{14}^{(3)} e_{14}$. Hence the first equality follows, and the remaining ones follow in a similar way.

Now we prove that there is a $K$-algebra isomorphism $R_q \cong KQ/\Omega_q$, where $Q = \mathcal{Q}(R_q)$ and $\Omega_q$ is the two-sided ideal of the path $K$-algebra $KQ$ of $Q$ generated by the following relations:

- $\beta_{j+1\,j}\beta_{j\,j+1}$ and $\beta_{j\,j+1}\beta_{j+1\,j}$, for $j = 1, \ldots, 5$, where $j+1$ is reduced modulo 5.
- $\beta_1\beta_2\beta_3\beta_4$, if there is a path $\bullet \xrightarrow{\beta_1} \bullet \xrightarrow{\beta_2} \bullet \xrightarrow{\beta_3} \bullet \xrightarrow{\beta_4} \bullet$ in $Q$.
- $\beta_{21}\beta_{15}\beta_{54}$, $\beta_{32}\beta_{21}\beta_{15}$, $\beta_{43}\beta_{32}\beta_{21}$, $\beta_{54}\beta_{43}\beta_{32}$, $\beta_{15}\beta_{54}\beta_{43}$;
- $q_{14}^{(5)}\beta_{12}\beta_{23}\beta_{34} - q_{13}^{(2)} q_{14}^{(3)}\beta_{15}\beta_{54}$,
- $q_{25}^{(1)}\beta_{23}\beta_{34}\beta_{45} - q_{24}^{(3)} q_{25}^{(4)}\beta_{21}\beta_{15}$,
- $q_{31}^{(2)}\beta_{34}\beta_{45}\beta_{51} - q_{35}^{(4)} q_{31}^{(5)}\beta_{32}\beta_{21}$,

- $q_{42}^{(3)}\beta_{45}\beta_{51}\beta_{12} - q_{41}^{(5)}q_{42}^{(1)}\beta_{43}\beta_{32}$,
- $q_{53}^{(4)}\beta_{51}\beta_{12}\beta_{23} - q_{52}^{(1)}q_{53}^{(2)}\beta_{54}\beta_{43}$.

It is easy to check that the correspondences $\varepsilon_j \mapsto e_j$ and $\beta_{ij} \mapsto e_{ij}$ define a $K$-algebra homomorphism $h : KQ/\Omega_q \to R_q$, where $\varepsilon_j$ is the primitive idempotent of the path algebra $KQ$ defined by the stationary path at the vertex $j$, for every $j \in (Q_\mu)_0$. Note that the map $h$ is well defined and surjective. Finally, since $\dim_K KQ/\Omega_q = \dim_K R_q = 25$, the surjection $h$ is an isomorphism of $K$-algebras.

Now it is easy to see that $R_q/\mathrm{soc}(R_q) \cong R_{\overline{q}}/\mathrm{soc}(R_{\overline{q}})$ and the algebra $KQ/\Omega_q \cong R_q$ is special biserial; hence $R_q$ is representation-tame, see [3, 5.2]. Since there is a cyclic walk

$$1 \xrightarrow{\beta_{12}} 2 \xrightarrow{\beta_{23}} 3 \xleftarrow{\beta_{43}} 4 \xrightarrow{\beta_{45}} 5 \xleftarrow{\beta_{51}} 1$$

of the quiver $Q$ and, according to the finite representation type criterion in [13], the algebra $R_q$ is of infinite representation type, see also [10, Proposition 3.7].

**Problem 4.5.** Give a characterisation of the Frobenius $\mathbb{A}$-full matrix algebras $R_{\mathbb{A}}$ such that $\dim_K R_{\mathbb{A}} = n^2$, $n \geq 3$, $\mathrm{soc}\, R_{\mathbb{A}} = J(R_{\mathbb{A}})^{n-2}$ and $J(R_{\mathbb{A}})^{n-1} = 0$.

**Remark 4.6.** In connection with Problem 4.5, we recall that if $R_{\mathbb{A}}$ is $\mathbb{A}$-full matrix algebra and $R_{\overline{\mathbb{A}}}$ is the $(0,1)$-limit of $R_{\mathbb{A}}$ then

- $J(R_{\mathbb{A}})^s = J(R_{\overline{\mathbb{A}}})^s$, for each $s \geq 1$ (by [7, Proposition 3.2]),
- $\mathrm{soc}\, R_{\mathbb{A}} = \mathrm{soc}\, R_{\overline{\mathbb{A}}}$ (by [7, Proposition 5.1]), and
- $R_{\mathbb{A}}$ is a Frobenius algebra if and only if the $(0,1)$-limit $R_{\overline{\mathbb{A}}}$ of $R_{\mathbb{A}}$ is a Frobenius algebra (by [7, Theorem 5.3]).

It follows that a solution of the Problem 4.5 for $(0,1)$-structure systems should help to find a solution for arbitrary structure systems $\mathbb{A}$.

We recall from [5] that in case $n = 5$, a list $(0,1)$-structure systems $\mathbb{A}$ such that $R_{\mathbb{A}}$ is a Frobenius algebra is given in Examples 4.7(4) and 4.7(5) of [5]. It is shown there that, up to isomorphisms of the $\mathbb{A}$-full matrix algebras, there are precisely four Frobenius $(0,1)$-structure systems $\mathbb{A}$. Note that one of them has the property $\mathrm{soc}\, R_{\mathbb{A}} = J(R_{\mathbb{A}})^3$, compare with the $(0,1)$-limit algebra $R_{\overline{q}}$ in Example 4.4.

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Received by the editors: ????
and in final form ????.