

Nilpotent subsemigroups of a semigroup of order-decreasing transformations of a rooted tree

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ABSTRACT. This paper deals with a semigroup of order-decreasing transformations of a rooted tree. Such are the transformations α of some rooted tree G satisfying following condition: for any x from G $\alpha(x)$ belongs to a simple path from x to the root vertex of G . We describe all subsemigroups of the mentioned semigroup, which are maximal among nilpotent subsemigroups of nilpotency class k in our semigroup. In the event when rooted tree is a ray we prove that all these maximal subsemigroups are pairwise nonisomorphic.

Introduction

Let T be a rooted tree with a natural partial order defined on the set of vertices (i.e. $x < y$ if x belongs to a simple path from y to the root of the tree). Let \mathcal{T}_T be a symmetric semigroup of all transformations of set of vertices of the rooted tree T . We do transformation from left to right, i.e. $(\varphi \cdot \psi)(x) = \psi(\varphi(x))$. A transformation $\alpha \in \mathcal{T}_T$ is called an order-decreasing transformation if for any x from T an inequality $\alpha(x) \leq x$ holds. It is easy to see that the set D_T of all order-decreasing transformations from \mathcal{T}_T forms a semigroup. In case of T is a finite chain this semigroup is called D_n . The semigroup D_n has been studied by many algebraists. Being introduced in Pin's monograph([4]) in connection with some problems of formal languages it was later considered by Howie at

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his lectures given in the University of Lisbon on combinatoric and arithmetical problems of the theory of transformation semigroup (some combinatoric results on D_n can be viewed in [3]) and also by Higgings. Umar wrote a series of papers (see, e.g. [5], [6]), investigating ideals, Rees congruences, idempotent rank and Green relations on D_n . More general semigroup of all contraction endomorphisms of arbitrary finite graph was considered by Vernitskii ([7]).

As D_T contains a zero 0, a transformation mapping all the vertices into the root, a question on study of non-trivial nilpotent subsemigroups from D_T naturally arises. For any mapping s from some nilpotent subsemigroup of D_T , we name by domain of s (*doms*) the set of vertices, which s does not map into the root; by the range of s (*rans*) we name the set of non-root vertices from $s(T)$; and by the rank of s we name the number of elements of *rans*. Let $Nil(T, k)$ denote the set of subsemigroups from D_T , which are maximal among nilpotent subsemigroups from D_T of nilpotency class k . The case when the tree is a finite chain was investigated in [8]. In our paper we describe all the semigroups from $Nil(T, k)$, and prove that all these semigroups are pairwise nonisomorphic in case when rooted tree T is a ray. Proving that we used the method of matching of nilpotent subsemigroups of the transformations semigroup to special partially ordered sets, this method first appeared in [2] and is explicitly described in [1].

1. The structure of maximal nilpotent subsemigroups from D_T

Let m be a vertex of T and A be a subset of the set of all vertices of T and $m \notin A$. Then we denote by $Less(m, A)$ the set of all vertices from A less than m ; by $Up(m, A)$ we denote the set of all vertices from A greater than m . By $less(m, A)$ and $up(m, A)$ we denote cardinalities of sets $Less(m, A)$ and $Up(m, A)$ correspondingly; by the $Less(m)$ and $Up(m)$ the sets $Less(m, T \setminus \{m\})$ and $Up(m, T \setminus \{m\})$ correspondingly. We fix some natural k less than the number of vertices of T and define $\Lambda(T, k)$ as a set of ordered partitions (i.e. with defined order of blocks (subsets)) of the non-root vertices of T into k nonempty non-overlapping blocks Q_1, \dots, Q_k , such that

$$\forall 1 \leq i < k, \quad \forall l \in Q_i \quad \exists m \in Q_{i+1} \quad m < l; \quad (1.1)$$

and $\forall 1 < i \leq k, \quad \forall h \in Q_i$

$$(Q_i \cap Less(h) \neq \emptyset) \Rightarrow (\exists l_1 \in Q_1, \dots, \exists l_{i-1} \in Q_{i-1} \quad l_1 > \dots > l_{i-1} > h) \quad (1.2)$$

Let's denote the root of tree T as r . For some partition λ from $\Lambda(T, k)$ with blocks Q_1, \dots, Q_k we consider

$$\mathcal{T}_\lambda = \{\varphi \in D_T \mid \forall m \leq k, \forall i \in Q_m \quad \varphi(i) \in (Q_{m+1} \cup \dots \cup Q_k \cup \{r\}) \cap \text{Less}(i)\}.$$

It is easy to verify that \mathcal{T}_λ is a subsemigroup from D_T .

Lemma 1. $\mathcal{T}_\lambda \in \text{Nil}(T, k)$.

Proof. For any $\varphi_1, \dots, \varphi_k$ from \mathcal{T}_λ and for any non-root vertex i from T we have:

$$\begin{aligned} \varphi_1(i) \in Q_2 \cup \dots \cup Q_k \cup \{r\}; \quad \varphi_2(\varphi_1(i)) \in Q_3 \cup \dots \cup Q_k \cup \{r\}; \quad \dots; \\ \varphi_k(\varphi_{k-1}(\dots \varphi_1(i) \dots)) \in \{r\}. \end{aligned}$$

Therefore \mathcal{T}_λ is nilpotent of nilpotency class not greater than k . Simultaneously, one can choose $k - 1$ elements from \mathcal{T}_λ , such that their product is not equal to zero. (e.g., one can select $\varphi_1^*, \dots, \varphi_{k-1}^*$, such that for some l_1 from Q_1 $\varphi_1^*(l_1) = l_2 \in Q_2$, $\varphi_2^*(l_2) = l_3 \in Q_3, \dots, \varphi_{k-1}^*(l_{k-1}) = l_k \in Q_k$, $l_k \neq r$. The existence of $l_2 \in Q_2, \dots, l_k \in Q_k$ such that $l_1 < l_2 < \dots < l_k$, follows from the definition of λ , (1.1). Then $\varphi_1^* \cdot \varphi_1^* \cdot \dots \cdot \varphi_{k-1}^* \neq 0$).

Hence we have that \mathcal{T}_λ is of nilpotency class k . Now we show the maximality of \mathcal{T}_λ . Indeed, let \mathcal{T}_λ be contained in some semigroup \mathcal{T} from $\text{Nil}(T, k)$ and $\mathcal{T} \neq \mathcal{T}_\lambda$. We consider ψ from $\mathcal{T} \setminus \mathcal{T}_\lambda$. Then there exist block Q_m and vertex $i \in Q_m$, such that $\xi = \psi(i)$ belongs to $Q_1 \cup \dots \cup Q_m$. From (1.1) it follows that

there exists $\varphi_1 \in \mathcal{T}_\lambda \setminus \{0\}$ such that $\varphi_1(\xi) \in Q_{m+1}$;

there exists $\varphi_2 \in \mathcal{T}_\lambda \setminus \{0\}$ such that $\varphi_2(\varphi_1(\xi)) \in Q_{m+2}$;

\dots ;

there exists $\varphi_{k-m} \in \mathcal{T}_\lambda \setminus \{0\}$ such that $\varphi_{k-m}(\dots \varphi_1(\xi) \dots) \in Q_k$.

Next, if $m = 1$ then $\psi \cdot \varphi_1 \cdot \dots \cdot \varphi_{k-m}(i) \in Q_k$, otherwise from $\xi \in Q_1 \cup \dots \cup Q_m$ and (1.1) it follows that $Q_m \cap \text{Less}(i) \neq \emptyset$. Then

there exist $\psi_1 \in \mathcal{T}_\lambda$, $i_1 \in Q_{m-1}$, such that $\psi_1(i_1) = i$;

there exist $\psi_2 \in \mathcal{T}_\lambda$, $i_2 \in Q_{m-2}$; such that $\psi_2(i_2) = i_1$;

\dots ;

there exist $\psi_{m-1} \in \mathcal{T}_\lambda$, $i_{m-1} \in Q_1$, such that $\psi_{m-1}(i_{m-1}) = i_{m-2}$.

Then $\psi_1 \cdot \psi_2 \cdot \dots \cdot \psi_{m-1} \cdot \psi \cdot \varphi_1 \cdot \dots \cdot \varphi_{k-m}(i_{m-1}) \in Q_k$. So, we have come to contradiction with the condition $\mathcal{T} \in \text{Nil}(T, k)$. \square

Let \mathcal{T} be a semigroup from $\text{Nil}(T, k)$. We define partial order $<^{\mathcal{T}}$ on T as following:

$$i <^{\mathcal{T}} j \Leftrightarrow \exists \varphi \in \mathcal{T} \quad \varphi(j) = i.$$

Since \mathcal{T} is a subsemigroup from D_T then obviously $i <^{\mathcal{T}} j$ implies $i < j$. For some vertex m from T let $\text{Less}_{\mathcal{T}}(m)$ stand for the set of all vertices

j from T such that $j <^T m$. Next we consider following sets

$$P_1 = \{i \in T \setminus \{r\} \mid \text{Less}_{\mathcal{T}}(i) = \{r\}\};$$

$$P_2 = \{i \in T \setminus (\{r\} \cup P_1) \mid \text{Less}_{\mathcal{T}}(i) \subset (P_1 \cup \{r\})\};$$

$$P_3 = \{i \in T \setminus (\{r\} \cup P_1 \cup P_2) \mid \text{Less}_{\mathcal{T}}(i) \subset (P_1 \cup P_2 \cup \{r\})\};$$

...

$$P_p = \{i \in T \setminus (\{r\} \cup P_1 \cup \dots \cup P_{p-1}) \mid \text{Less}_{\mathcal{T}}(i) \subset (P_1 \cup P_2 \dots \cup P_{p-1} \cup \{r\})\};$$

...

Obviously, $P_1 \cup \dots \cup P_p \cup \dots = T \setminus \{r\}$. Let p_{max} be the greatest among indices p , for which $P_p \neq \emptyset$. From the fact, that \mathcal{T} is a semigroup of nilpotency class k we conclude that $p_{max} = k$. What is more, for any i, j less than k we have that $P_i \cap P_j = \emptyset$.

Sets $Q_1 = P_{p_{max}}, \dots, Q_{p_{max}} = P_1$ form a partition of set $T \setminus \{1\}$ written in following as $\lambda_{\mathcal{T}}$. (3)

Lemma 2. *For any semigroup $\mathcal{T} \in \text{Nil}(T, k)$, partition $\lambda_{\mathcal{T}}$ belongs to $\Lambda(T, k)$.*

Proof. Let i be from Q_l , $l < k$. Then there exist $\varphi \in \mathcal{T}$, $j \in Q_{l+1}$, such that $\varphi(i) = j$; hence $j < i$. Thus $\lambda_{\mathcal{T}}$ meets the requirement (1.1). Next, if for all $w > 1$ there are no vertices m and l from the block Q_w such that $m < l$, then $\lambda_{\mathcal{T}}$ satisfies condition (1.2). Now, let $m, h \in Q_w$ and $h < m$. We take a transformation φ with $\text{dom}\varphi = m$, $\text{ran}\varphi = h$. It is easy to see that φ belongs to $D_{\mathcal{T}}$. Let there be no sequence $q_1 \in Q_1, \dots, q_{w-1} \in Q_{w-1}$ satisfying $q_1 > \dots > q_{w-1} > m$. Then for any $\phi_1, \phi_2, \dots, \phi_k$ elements from the semigroup $\langle \mathcal{T}, \varphi \rangle$ (obtained from adjoining φ to \mathcal{T}) it is true that $\phi_1 \cdot \phi_2 \cdot \dots \cdot \phi_k = 0$. Since \mathcal{T} is maximal, we have come to the contradiction. Thus (1.2) must be satisfied and $\lambda_{\mathcal{T}}$ belongs to $\Lambda(T, k)$. \square

Theorem 1. *There are reciprocal mappings φ and ψ which set up one-to-one correspondence between $\Lambda(T, k)$ and $\text{Nil}(T, k)$ defined as follows*

1. $\varphi : \Lambda(T, k) \rightarrow \text{Nil}(T, k)$, $\forall \lambda \in \Lambda(T, k)$ $\varphi(\lambda) = \mathcal{T}_{\lambda}$ (see (1.1)-(1.2))
2. $\psi : \text{Nil}(T, k) \rightarrow \Lambda(T, k)$, $\forall \mathcal{T} \in \text{Nil}(T, k)$ $\psi(\mathcal{T}) = \lambda_{\mathcal{T}}$ (see (3)).

Proof. Let λ be a partition from $\Lambda(T, k)$ with blocks Q_1, \dots, Q_k . We consider $\psi(\varphi(\lambda)) = \psi(\mathcal{T}_{\lambda})$. It is a partition from $\Lambda(n, k)$ with blocks $Q_1^{\psi}, \dots, Q_k^{\psi}$. For any j from the block Q_k we have: $\text{Less}_{\mathcal{T}_{\lambda}}(j) = \{r\}$. Therefore $Q_k \subset Q_k^{\psi}$. Now we take an arbitrary i from $T \setminus Q_k$. From the

definition of \mathcal{T}_λ it follows that there exists β from \mathcal{T}_λ , such that $i \in \text{dom}\beta$. Therefore $\text{Less}_{\mathcal{T}_\lambda}(i) \neq \{r\}$ and $Q_k = Q_k^\psi$. Next, for any j from Q_{k-1} we have: $\text{Less}_{\mathcal{T}_\lambda}(j) \subset Q_k \cup \{r\} = Q_k^\psi \cup \{r\}$. Therefore $Q_{k-1} \subset Q_{k-1}^\psi$. Now we take an arbitrary i from $T \setminus (Q_k \cup Q_{k-1})$. From the definition of \mathcal{T}_λ it follows that there exists β from \mathcal{T}_λ , such that $i \in \text{dom}\beta$ and $\beta(i) \in Q_{k-1}$. Therefore $i \notin Q_{k-1}^\psi$ and $Q_{k-1} = Q_{k-1}^\psi$.

Further we move by induction. Let an equality $Q_{k-l+1} = Q_{k-l+1}^\psi$ be held. For any i from Q_{k-l} an inclusion $\text{Less}_{\mathcal{T}_\lambda}(i) \subset Q_k \cup \dots \cup Q_{k-l+1} \cup \{r\}$ holds. Simultaneously, for any i from $T \setminus (Q_k \cup \dots \cup Q_{k-l})$ there exists β from \mathcal{T}_λ , such that $i \in \text{dom}\beta$ and $\beta(i) \in Q_{k-l}$. Therefore $Q_{k-l} = Q_{k-l}^\psi$. Hence $\psi(\varphi(\lambda)) = \lambda$. Now let us take some \mathcal{T} from $\text{Nil}(T, k)$. We consider $\varphi(\psi(\mathcal{T})) = \varphi(\lambda_{\mathcal{T}})$. It is a semigroup from $\text{Nil}(T, k)$. $\lambda_{\mathcal{T}}$ is a partition from $\Lambda(T, k)$ with blocks Q_1, \dots, Q_k . Let us take some α from \mathcal{T} . For any element j from the block Q_k we have that $\alpha(j) = r$ (for the definition of $\lambda_{\mathcal{T}}$). Let now $j \in Q_{k-1}$. Then $\alpha(j)$ belongs to the set $Q_k \cup \{r\}$ (for the definition of $\lambda_{\mathcal{T}}$). Next, for the definition of $\lambda_{\mathcal{T}}$ for any element i from the block Q_{k-l} we have that $\alpha(i)$ does not belong to any of the sets Q_1, Q_2, \dots, Q_{k-l} . Hence we get that \mathcal{T} belongs to $\varphi(\lambda_{\mathcal{T}})$. As \mathcal{T} is maximal among the nilpotent subsemigroups from D_n of nilpotency class k , then $\mathcal{T} = \varphi(\psi(\mathcal{T})) = \varphi(\mathcal{T}_\lambda)$. \square

2. Equivalence relations \sim^R and \sim^L and their properties

Here and in the following we consider the case when T is a ray. Surely, one can number vertices in such a way that T is isomorphic to the set of natural numbers \mathbb{N} . Let's define relations \sim^R and \sim^L on some $\mathcal{T} \in \text{Nil}(\mathbb{N}, k)$ as follows: for any elements x, y from $\mathcal{T} \in \text{Nil}(\mathbb{N}, k)$

1. $x \sim^R y \Leftrightarrow$ for all $t \in \mathcal{T}$ $tx = ty$;
2. $x \sim^L y \Leftrightarrow$ for all $t \in \mathcal{T}$ $xt = yt$.

It is easy to prove that \sim^R and \sim^L are equivalence relations.

Proposition 1. 1. $x \sim^R y \Leftrightarrow \forall m \in \mathbb{N} \setminus Q_1 \quad (x(m) = y(m))$;

2. $x \sim^L y \Leftrightarrow \forall m \in \mathbb{N}$ if $x(m) \in \bigcup_{i=1}^{k-1} Q_i$ then $x(m) = y(m)$,
if $x(m) \in Q_k$ then $y(m) \in Q_k$.

Proof. Let x and y be from the semigroup \mathcal{T} and $\forall t \in \mathcal{T} : tx = ty$. For any m from $\mathbb{N} \setminus Q_1$ let s be from \mathcal{T} such that $rans = m$. Then $sx = sy$ implies $x(m) = y(m)$.

Simultaneously, let x and y be from \mathcal{T} and for any m from $\mathbb{N} \setminus Q_1$ $x(m) = y(m)$. Let's take an arbitrary element s from \mathcal{T} . Then $\text{dom}sx = \text{dom}sy$ and $\text{dom}sx \in \mathbb{N} \setminus Q_1$, so $sx = sy$ and $x \sim^R y$. The second part of the proposition can be proved analogously. \square

Corollary 1. *Let \mathcal{T} be a semigroup from $\text{Nil}(\mathbb{N}, k)$ with the correspondent partition λ from $\Lambda(\mathbb{N}, k)$. Then all the blocks of the partition λ except Q_1 are finite if and only if the number of equivalence classes generated by the equivalence relation \sim^R on the semigroup \mathcal{T} is finite.*

Proof. It is obvious that if at least one of the blocks Q_2, \dots, Q_k is infinite, then \mathcal{T} has infinite number of equivalence classes for the relation \sim^R . Simultaneously, if all the blocks Q_2, Q_3, \dots, Q_k are finite then the number of equivalence classes is also finite and equals

$$\prod_{m \in Q_i, 2 \leq i \leq k} (\text{less}(m, Q_{i+1} \cup \dots \cup Q_k) + 1).$$

\square

3. Non-isomorphism theorem

Theorem 2. *Let $k > 2$. Then all semigroups from $\text{Nil}(\mathbb{N}, k)$ are pairwise non-isomorphic.*

Proof. We show that it is possible to restore the correspondent partition λ from $\Lambda(\mathbb{N}, k)$ from the properties of an arbitrary semigroup from $\text{Nil}(\mathbb{N}, k)$ as an abstract semigroup. To do this, we use induction for nilpotency class k . First we consider the case of $k = 3$. Let \mathcal{T} be a semigroup from $\text{Nil}(\mathbb{N}, 3)$ with a correspondent partition $\lambda \in \Lambda(\mathbb{N}, 3)$, which consists of the blocks

$$\begin{aligned} Q_1 &= \{\dots, a_i, \dots, a_2, a_1\} & (a_1 < a_2 < \dots < a_i < \dots), \\ Q_2 &= \{\dots, b_i, \dots, b_2, b_1\} & (b_1 < b_2 < \dots < b_i < \dots), \\ Q_3 &= \{\dots, c_i, \dots, c_2, c_1\} & (c_1 < c_2 < \dots < c_i < \dots). \end{aligned}$$

Let's show that

$$\min_{s \in \mathcal{T}, |s\mathcal{T}| \neq 1, |s\mathcal{T}| < \infty} |s\mathcal{T}| = \text{less}(b_1, Q_3) + 1.$$

Indeed, if $|s\mathcal{T}| \neq 1$, then there exist a and b ($a \in Q_1, b \in Q_2$) such that $s(a) = b$. Then for any $c \in Q_3 \cap \text{Less}(b, Q_3)$ an ideal $s\mathcal{T}$ contains an element of rank 1 that maps a into c . Therefore $|s\mathcal{T}| \geq \text{less}(b, Q_3) + 1 \geq$

$less(b_1, Q_3) + 1$.

On the other hand, \mathcal{T} contains a mapping of range b_1 . At the same time, $|s_0\mathcal{T}| = less(b_1, Q_3) + 1$. Therefore we have:

$$\min_{s \in \mathcal{T}, |s\mathcal{T}| \neq 1, |s\mathcal{T}| < \infty} |s\mathcal{T}| = less(b_1, Q_3) + 1.$$

For each subset A of Q_1 there is a right ideal $s\mathcal{T}$, satisfying $|s\mathcal{T}| = less(b_1, Q_3) + 1$:

$$s\mathcal{T} = \{ \varphi \in \mathcal{T}, |ran\varphi| = 1, dom\varphi = A, ran\varphi \in Less(b_1, Q_3) \},$$

and there is no other ideal $s\mathcal{T}$ of cardinality $less(b_1, Q_3) + 1$. We denote the set of such ideals by Θ_1 . By B_1 we stand for the set of all numbers b of Q_2 , for which $less(b, Q_3) = less(b_1, Q_3)$. Next, let $W = \{s \in \mathcal{T} | \text{if for some } t_1 \in \mathcal{T} \ st_2 \neq 0 \text{ and } t_1t_2 \neq 0 \text{ then for all } t_3 \in \mathcal{T}, st_3 \neq 0 \Rightarrow t_1t_3 \neq 0\}$.

It is easy to verify that for all s from W $|rans \cap Q_2| = 1$.

For any set X from Θ_1 we consider the number of equivalence classes of \sim^L on the set of elements s from W such that $s\mathcal{T} = X$. If $s\mathcal{T} \in \Theta_1$ for some s of W , then there exists only one b of Q_2 , which belongs to $rans$, and $b \in B_1 \cap Less\left(\min_{a \in s^{-1}(b)} a, Q_2\right)$, as $B_1 \cap \left(\bigcap_{a \in s^{-1}(b)} Less(a, Q_2)\right) = B_1 \cap$

$Less\left(\min_{a \in s^{-1}(b)} a, Q_2\right)$ holds. Using proposition 1 we can conclude that

among the numbers $|\{s \in W, s\mathcal{T} = X\} / \sim^L|$ of equivalency classes for the relation \sim^L on the set $\{s \in W | s\mathcal{T} = X\}$ where $X \in \Theta_1$ one can find only the numbers $|B_1|$ and $|less(a, Q_2)|$, where $|less(a, Q_2)| < |B_1|$, $a \in Q_1$. Hence we can say whether the set B_1 is finite or not. Next, from the abstract properties of \mathcal{T} we can get numbers

$$less(b_1, Q_3) = \dots = less(b_{|B_1|}, Q_3), |B_1| \text{ and the set of numbers } \Omega_1 = \{\alpha_1, \dots, \alpha_{i_1}\} = \{less(a, Q_2) | a \in Less(b_{|B_1|}, Q_1)\}.$$

Now we consider

$$\Theta_2 = \left\{ s\mathcal{T} : \exists X \in \Theta_1, s\mathcal{T} = \bigcap_{\tau \in \mathcal{T}, \tau\mathcal{T} \cap \Theta_1 = X, X \neq \tau\mathcal{T}} \tau\mathcal{T} \right\}.$$

If Θ_2 is an empty set, then $B_1 = Q_2$. If $\Theta_2 \neq \emptyset$, then $Q_2 \setminus B_1 \neq \emptyset$ and for every X from Θ_2 equality $|X| = less(b_{|B_1|+1}, Q_3)$ holds. Indeed, if $X = s\mathcal{T}$ belongs to Θ_2 , then there exists only one element b from the second block of the partition λ , which belongs to $rans$, because otherwise X would have two different ideals from Θ_1 . It is easy to see that $b \in Q_2 \setminus B_1$. In such a case $|X| = less(b, Q_3) \geq less(b_{|B_1|+1}, Q_3)$. Let s_0 be

an element of rank 1 and $rans_0 = b_{|B_1|+1}$. Then $|s_0\mathcal{T}| = less(b_{|B_1|+1}, Q_3)$ and so $|X| = less(b_{|B_1|+1}, Q_3)$.

We define set B_2 as following:

$$B_2 = \left\{ b \in Q_2, less(b, Q_3) = less(b_{|B_1|+1}, Q_3) \right\}.$$

For any set X of Θ_2 we consider the number of equivalency classes for the relation \sim^L on the set of elements s from \mathcal{T} such that $s\mathcal{T} = X$. If $s\mathcal{T} \in \Theta_2$ for s of \mathcal{T} , then there exists only one b of Q_2 , which belongs to $rans$, and as $B_2 \cap \left(\bigcap_{a \in s^{-1}(b)} Less(a, Q_2) \right) = B_2 \cap Less\left(\min_{a \in s^{-1}(b)} a, Q_2 \right)$, then $b \in B_2 \cap \left(\bigcap_{a \in s^{-1}(b)} Less(a, Q_2) \right)$. Hence we conclude that among the numbers $|\{s \in \mathcal{T} | s\mathcal{T} = X\} / \sim^L|$ of equivalence classes for the relation \sim^L on the set $\{s \in \mathcal{T} | s\mathcal{T} = X\}$ for all $X \in \Theta_2$ there are numbers $|B_2|$ and $|less(a, Q_2)| - |B_1|$ (where $a \in (Less(b_{|B_2|+|B_1|}, Q_1) \setminus Less(b_{|B_1|}, Q_1))$) only. Hence for the general properties of semigroup \mathcal{T} we can say whether the set B_2 is finite or not. Next, we get numbers $less(b_{|B_1|+1}, Q_3) = \dots = less(b_{|B_1|+|B_2|}, Q_3)$, $|B_2|$ and the set of numbers $\Omega_2 = \{\alpha_1^2, \dots, \alpha_{i_2}^2\} =$

$$= \{less(a, Q_2) | a \in (Less(b_{|B_2|+|B_1|}, Q_1) \setminus Less(b_{|B_1|+1}, Q_1))\}$$

$$(\alpha_1^2 < \alpha_2^2 < \dots < \alpha_i^2 < \alpha_{i+1}^2 < \dots)$$

Then we define sets

$$\Theta_j = \{s\mathcal{T} \mid \exists X \in \Theta_{j-1}, s\mathcal{T} = \bigcap_{\substack{\tau \in \mathcal{T}, \\ \tau\mathcal{T} \cap (\bigcup_{m=1}^{j-1} \Theta_m) = X, \\ \tau\mathcal{T} \neq X}} \tau\mathcal{T}\};$$

$$B_j = \left\{ b \in Q_2, less(b, Q_3) = less(b_{|B_1|+\dots+|B_{j-1}|+1}, Q_3) \right\}.$$

If Θ_j is an empty set, then $\bigcup_{i=1}^{j-1} B_i = Q_2$ and the process of considering

Θ_j is finished. If not, then $Q_2 \setminus \bigcup_{i=1}^{j-1} B_i \neq \emptyset$ and it is easy to prove by induction that for each set X of Θ_j $|X| = less(b_{|B_1|+|B_2|+\dots+|B_{j-1}|+1}, Q_3)$. Indeed, if $X = s\mathcal{T}$ belongs to Θ_j , then there exists only one element b from the second block of the partition λ , which belongs to $rans$, because otherwise X has two different ideals from Θ_1 for some $i < j$.

It is clear that $b \in Q_2 \setminus \bigcup_{i=1}^{j-1} B_i$. In such a case $|X| = less(b, Q_3) \geq less(b_{|B_1|+|B_2|+\dots+|B_{j-1}|+1}, Q_3)$. Let s_0 be an element of rank 1 and $rans_0 =$

$b_{|B_1|+|B_2|+\dots+|B_{j-1}|+1}$. Then

$$|s_0\mathcal{T}| = \text{less}(b_{|B_1|+|B_2|+\dots+|B_{j-1}|+1}, Q_3)$$

and so $|X| = \text{less}(b_{|B_1|+|B_2|+\dots+|B_{j-1}|+1}, Q_3)$.

For each set X of Θ_j we consider number of equivalency classes for the relation \sim^L on the set $\{s \in \mathcal{T} \mid s\mathcal{T} = X\}$. If $s\mathcal{T} \in \Theta_j$ for some s from \mathcal{T} , then there exists only one b of Q_2 which belongs to $\text{dom}s$,

and at that $b \in B_j \cap \left(\bigcap_{a \in s^{-1}(b)} \text{Less}(a, Q_2) \right) = B_j \cap \text{Less}\left(\min_{a \in s^{-1}(b)} a, Q_2 \right)$.

Hence we have that among the numbers of equivalency classes for the relation \sim^L on the set $\{s \in \text{Ann}_L\mathcal{T} \mid s\mathcal{T} = X\}$, where X is an element of Θ_j , there are numbers $|B_j|$ and $|\text{less}(a, Q_2)| - \sum_{i=1}^{j-1} |B_i|$ (where

$a \in (\text{Less}(b_{|B_1|+\dots+|B_j|}, Q_1) \setminus \text{Less}(b_{|B_1|+\dots+|B_{j-1}|}, Q_1))$) only. Hence we can say whether the set B_j is finite or not. So, for the general properties of \mathcal{T} we can obtain numbers $\text{less}(b_{|B_1|+\dots+|B_{j-1}|+1}, Q_3) = \dots = \text{less}(b_{|B_1|+\dots+|B_j|}, Q_3)$, $|B_j|$ and the set of numbers $\Omega_j = \{\alpha_1^j, \dots, \alpha_{i_j}^j\} =$

$$= \left\{ \text{less}(a, Q_2) - \sum_{i=1}^{j-1} |B_i| : a \in \text{Less}(b_{|B_1|+\dots+|B_j|}, Q_1) \setminus \text{Less}(b_{|B_1|+\dots+|B_{j-1}|}, Q_1) \right\}$$

At last we have next sets of numbers:

1. $\{\text{less}(b, Q_3), b \in Q_2\}$
2. $\{|B_1|, \dots, |B_i|, \dots\}$
3. $\Omega_j = \left\{ \text{less}(a, Q_2) - \sum_{i=1}^{j-1} |B_i| : a \in \text{Less}\left(b_{\left(\sum_{i=1}^j |B_i|\right)}, Q_1\right) \setminus \text{Less}\left(b_{\left(\sum_{i=1}^{j-1} |B_i|\right)}, Q_1\right) \right\}$.

It is clear that $\forall a \in Q_1$ either there exists α_m^l from some Ω_m such that $\text{less}(a, Q_2) = \alpha_m^l + |B_1| + \dots + |B_{m-1}|$ or there exists such j that $\text{less}(a, Q_2) = |B_j|$.

Let's now consider such ideals X from Θ_1 , for which

1. number of equivalence classes for the relation \sim^L on the set $\{s \in \mathcal{T}, s\mathcal{T} = X\}$ equals α_2^1 — the next to the least number of Ω_1 ;
2. the set $\{s \in \text{Ann}_L\mathcal{T}, s\mathcal{T} = X\}$ is finite.

To each such ideal we conform a number $|\{s \in \text{Ann}_L \mathcal{T}, s\mathcal{T} = X\}|$. Considering all sets X satisfying 1-2, we get some set of numbers $|\{s \in \text{Ann}_L \mathcal{T}, s\mathcal{T} = X\}|$ with repetition, which we denote by $\Psi_{\alpha_1^1}$ (under a set with a repetition we mean a set where each number has it's repetition factor). If number of equivalence classes for the relation \sim^L on the set $\{s \in \mathcal{T}, s\mathcal{T} = X\}$ is equal to α_2^1 , then for $A_{\alpha_1^1}^1 = \{a \in Q_1 : \text{less}(a, Q_2) = \alpha_1^1\}$, $((s \in \text{Ann}_L \mathcal{T}) \wedge (s\mathcal{T} = X))$ implies $s(A_{\alpha_1^1}^1) \subset Q_3$. Also for some a from Q_1 $\text{less}(a, Q_2) < |B_1|$ implies $\text{Less}(a, Q_3) = \text{Less}(b_{k_2}, Q_3)$ and therefore $\text{less}(a, Q_3) = \text{less}(b_{k_2}, Q_3)$. Thus the least element of $\Psi_{\alpha_1^1}$ is

the number $\alpha_2^1 \left(\text{less}(b_{k_2}, Q_3) + 1 \right)^{|A_{\alpha_1^1}^1|}$. Hence we can get $|A_{\alpha_1^1}^1|$ from the

general properties of semigroup \mathcal{T} . Now let's consider ideals X from Θ_1 such that the number of equivalence classes for the relation \sim^L on the set $\{s \in \text{Ann}_L \mathcal{T}, s\mathcal{T} = X\}$ equals α_3^1 — the number from Ω_1 , next to α_2^1 , and the set $\{s \in \text{Ann}_L \mathcal{T}, s\mathcal{T} = X\}$ is finite. To each such ideal we conform the number $|\{s \in \text{Ann}_L \mathcal{T}, s\mathcal{T} = X\}|$, considering all such X , we get some set of natural numbers $|\{s \in \text{Ann}_L \mathcal{T}, s\mathcal{T} = X\}|$ with repetition, which we denote by $\Psi_{\alpha_2^1}$. Let $A_{\alpha_2^1}^1 = \{a \in Q_1 : \text{less}(a, Q_2) = \alpha_2^1\}$. The

least element of $\Psi_{\alpha_2^1}$ is the number $\alpha_3^1 (\text{less}(b_{k_2}, Q_3) + 1)^{|A_{\alpha_1^1}^1| + |A_{\alpha_2^1}^1|}$. Hence we get the number $|A_{\alpha_2^1}^1|$ from the abstract properties of \mathcal{T} . Now let $A_{\alpha_i}^1 =$

$\{a \in Q_1 : \text{less}(a, Q_2) = \alpha_i^1\}$ for every α_i^1 of Ω_1 . Next we consider ideals X from Θ_1 such that the number of equivalence classes for the relation \sim^L on the set $\{s \in \mathcal{T}, s\mathcal{T} = X\}$ is equal to an element α_i^1 from Ω_1 , and the set $\{s \in \text{Ann}_L \mathcal{T}, s\mathcal{T} = X\}$ is finite. To each such ideal we conform a number $|\{s \in \text{Ann}_L \mathcal{T}, s\mathcal{T} = X\}|$; taking all such X , we get some set of natural numbers $|\{s \in \text{Ann}_L \mathcal{T}, s\mathcal{T} = X\}|$ with repetition, which we

denote by $\Psi_{\alpha_i^1}$. The least element of $\Psi_{\alpha_i^1}$ is $\alpha_3^1 \left(\text{less}(b_{k_2}, Q_3) + 1 \right)^{\sum_{l=1}^{i-1} |A_{\alpha_l}^1|}$.

Therefore we can get $|A_{\alpha_{i-1}}^1|$ from the general properties of our semigroup.

Let $A_{|B_1|}$ denote the set of all a from Q_1 such that $\text{less}(a, Q_2) = |B_1|$.

Now let's assume that the set B_1 is finite. We investigate ideals X of

Θ_1 for which equivalence classes for the relation \sim^L on the set $\{s \in \text{Ann}_L \mathcal{T}, s\mathcal{T} = X\}$ equals $|B_1|$, and the set $\{s \in \text{Ann}_L \mathcal{T}, s\mathcal{T} = X\}$ is

finite. To each such ideal we conform the number $|\{s \in \text{Ann}_L \mathcal{T}, s\mathcal{T} = X\}|$. We get the set of numbers $|\{s \in \text{Ann}_L \mathcal{T}, s\mathcal{T} = X\}|$ with repetition,

let's denote it by Γ . Clearly, each mapping s from the left annihilator \mathcal{T} ,

for which $s\mathcal{T} = X$, maps some nonempty subset from $Q_1 \setminus \bigcup_{\alpha_i^1 \in \Omega_1} A_{\alpha_i}^1$ into

an element from B_1 , and all the other elements from Q_1 — into elements

from $Q_3 \cup \{1\}$. At that s maps elements $a \in \bigcup_{\alpha \in \Omega_1} A_\alpha$ into $Q_3 \cup \{1\}$, and at least one element of $A_{|B_1|}$ must be mapped into Q_2 . Therefore Γ contains numbers of type $|B_1| \left(\prod_{a \in A} (\text{less}(a, Q_3) + 1) \right)$, where $A \subset Q_1$, $A \cap A_{|B_1|} \neq A_{|B_1|}$, and $\bigcup_{\alpha_i^1 \in \Omega_1} A_{\alpha_i^1} \subset A$. The least element among all elements of Γ

is $|B_1| \left(\text{less}(b_{k_2}, Q_3) + 1 \right)^{\sum_{\alpha_i^1 \in \Omega_1} |A_{\alpha_i^1}|}$. Hence we get $|A_{\alpha_{i_1}}|$ (α_{i_1} is the greatest number of Ω_1). Let's denote the least element of Γ by ξ , and $\bigcup_{\alpha_i^1 \in \Omega_1} A_{\alpha_i^1}$ by A_{Ω_1} . Now we consider ideals X from Θ_2 such that number

of equivalence classes for the relation \sim^L on the set $\{s \in \text{Ann}_L \mathcal{T}, s\mathcal{T} = X\}$ is equal to the least element α_1^2 of Ω_2 ; and the set $\{s \in \text{Ann}_L \mathcal{T} : s\mathcal{T} = X\}$ is finite. To each such ideal we conform the number $|\{s \in \text{Ann}_L \mathcal{T}, s\mathcal{T} = X\}|$. We get some set of numbers with repetition, which we denote by $\Psi_{\alpha_1^2}$. If Θ_2 is empty, then $\Psi_{\alpha_1^2}$ is also empty and thus Q_2 is finite and for all $a \in Q_1$ $\text{less}(a, Q_3) = \text{less}(b_1, Q_3)$. If Θ_2 is not empty, then $\Psi_{\alpha_1^2}$ is not empty too, and the least element of $\Psi_{\alpha_1^2}$ is the number

$\alpha_1^2 \left(\text{less}(b_1, Q_3) + 1 \right)^{\sum_{\alpha \in \Omega_1} |A_\alpha^1|} \left(\prod_{a \in Q_1, \text{less}(s, Q_2) = |B_1|} (\text{less}(a, Q_3) + 1) \right)$, in

case of the set $A_{|B_1|} = \{a \in Q_1 : \text{less}(a, Q_2) = |B_1|\}$ is nonempty, and $\alpha_1^2 (\text{less}(b_1, Q_3) + 1)^{\sum_{\alpha \in \Omega_1} |A_\alpha^1|}$ otherwise. Hence we can say whether the set $A_{|B_1|}$ is empty, and if not we have the number

$$\prod_{a \in A_{|B_1|}} (\text{less}(a, Q_3) + 1).$$

Let

$$\eta = \begin{cases} \prod_{a \in A_{|B_1|}} (\text{less}(a, Q_3) + 1), & A_{|B_1|} = \emptyset; \\ 1, & A_{|B_1|} \neq \emptyset. \end{cases}$$

Let's remove one number ξ from Γ . Now the least element of Γ and the one next to it are $\xi \cdot (\text{less}(a_{|A_{\Omega_1|+1}}, Q_3) + 1)$ and $\xi \cdot (\text{less}(a_{|A_{\Omega_1|+2}}, Q_3) + 1)$.

So, we get numbers $\text{less}(a_{|A_{\Omega_1|+1}}, Q_3)$ and $\text{less}(a_{|A_{\Omega_1|+2}}, Q_3)$.

We remove the number $\xi \cdot (\text{less}(a_{|A_{\Omega_1|+2}}, Q_3) + 1)$ from Γ ;

if $\eta \neq (\text{less}(a_{|A_{\Omega_1|+1}}, Q_3) + 2)$ then we take away a number

$\xi \cdot \text{less}(a_{|A_{\Omega_1|+1}}, Q_3)$ from Γ .

Next, if $\eta \neq (\text{less}(a_{|A_{\Omega_1|+2}}, Q_3) + 1)(\text{less}(a_{|A_{\Omega_1|+1}}, Q_3) + 1)$ and

$\eta \neq (\text{less}(a_{|A_{\Omega_1|}}, Q_3) + 1)$, then we take away the number

$\xi \cdot (less(a_{|A_{\Omega_1|+1}, Q_3}) + 1)(less(a_{|A_{\Omega_1|+2}, Q_3}) + 1)$ from Γ .
 Now the least element of Γ is $\xi(less(a_{|A_{\Omega_1|+3}, Q_3}) + 1)$. So we get $less(a_{|A_{\Omega_1|+3}, Q_3})$. We remove next numbers from Γ

- $\xi(less(a_{|A_{\Omega_1|+3}, Q_3}) + 1)$;
- $\xi(less(a_{|A_{\Omega_1|+3}, Q_3}) + 1)(less(a_{|A_{\Omega_1|+1}, Q_3}) + 1)$;
- $\xi(less(a_{|A_{\Omega_1|+3}, Q_3}) + 1)(less(a_{|A_{\Omega_1|+1}, Q_3}) + 1)$,
 if $(less(a_{|A_{\Omega_1|+1}, Q_3}) + 1) \neq \eta$;
- $(less(a_{|A_{\Omega_1|+3}, Q_3})+1)(less(a_{|A_{\Omega_1|+1}, Q_3})+1)(less(a_{|A_{\Omega_1|+2}, Q_3})+1)$,
 if $(less(a_{|A_{\Omega_1|+1}, Q_3}) + 1)(less(a_{|A_{\Omega_1|+2}, Q_3}) + 1) \neq \eta$ and
 $(less(a_{|A_{\Omega_1|+3}, Q_3})+1)(less(a_{|A_{\Omega_1|+1}, Q_3})+1)(less(a_{|A_{\Omega_1|+2}, Q_3})+1) \neq \eta$.

Now the least element of Γ is $|B_1|(less(a_{|A_{\Omega_1|+4}, Q_3}) + 1)$. We remove each time the least element and it's products with already removed numbers from Γ . Gradually we obtain numbers $less(a, Q_3)$ for all numbers a of the first block of the partition.

Now let B_1 be an infinite set. Obviously, in such a case Q_3 must be finite and thus for every element a from Q_1 an equality $less(a, Q_3) = less(b_1, Q_3) + 1$ holds. Next, Ω_1 also is an infinite set and $Q_1 = \bigcup_{\alpha \in \Omega_1} A_\alpha$ (implies from the definition of the set $\Lambda(\mathbb{N}, k)$); considering minimal elements of described above sets $\Psi_{\alpha_i^1}(\alpha_i^1 \in \Omega_1)$ we can get cardinalities of sets $|A_{\alpha_i^1}|$.

For any natural n we denote by A_n the set

$$\{a \in A : less(a, Q_2) = n\}.$$

To each of sets Ω_j we add the number $\alpha_m^j = |B_j|$, $m = \max_{\alpha_i^j \in \Omega_j} i + 1$. Now

we have some set Ω'_j . We consider ideals X of Θ_j such that the number of equivalence classes for the relation \sim^L on the set $\{s \in Ann_L \mathcal{T}, s\mathcal{T} = X\}$ equals some $\alpha_i^j \in \Omega'_j$, and the set $\{s \in Ann_L \mathcal{T} : s\mathcal{T} = X\}$ is finite. To every such ideal we conform the number $|\{s \in Ann_L \mathcal{T}, s\mathcal{T} = X\}|$. Hence we get some set of numbers with repetition $\Phi_{\alpha_i^j}$ with the least element

$$\alpha_i^j \left(\prod_{a \in Q_1: less(a, Q_2) \leq |B_{j-1}|} (less(a, Q_3) + 1) \right) \left(less(b_{|B_1|+\dots+|B_{j-1}|+1}, Q_3) + 1 \right)^{\sum_{q=1}^{l-1} |A_{\alpha_q^j}|},$$

if α_i^j is not the least element of Ω'_j ; and

$$\alpha_i^j \cdot \left(\prod_{a \in Q_1: \text{less}(a, Q_2) \leq |B_{j-1}|} (\text{less}(a, Q_3) + 1) \right),$$

if α_i^j is the least element of Ω'_j . Hence we gradually get numbers $|A_{\alpha_i^j}|$ for all α_i^j from Ω_j . Now we divide the least element of the set $\Phi_{\alpha_m^{j-1}}$ by the least element of the set $\Phi_{\alpha_1^j}$ ($\alpha_m^{j-1} = \max_{\alpha \in \Omega'_{j-1}} \alpha$). If the obtained number equals 1, then the set $A_{|B_j|}$ is empty; otherwise we get the number $\prod_{a \in A_{|B_j|}} (\text{less}(a, Q_3) + 1)$. As we already know numbers $A_{|\alpha|}$ where $\alpha \in \Omega'_1 \cup \Omega_2$ and $\text{less}(a, Q_3)$ for each $a \in Q_1$, then we can find i such that the

obtained number $\prod_{a \in A_{|B_j|}} (\text{less}(a, Q_3) + 1)$ is equal to the number

$$\prod_{\substack{\sum_{\alpha \in \Omega'_1 \cup \Omega_2} |A_\alpha| + 1 \leq l \leq \sum_{\alpha \in \Omega'_1 \cup \Omega_2} |A_\alpha| + 1 + i}} \text{less}(a_l, Q_3).$$

So, we get $|A_{|B_j|}| = i$. Analogously we get numbers $|A_{|B_j|}|, j > 2$.

It is necessary to note that if at some step B_j is an infinite set, then it means that the block Q_3 is finite and $Q_1 = \bigcup_{\alpha \in \bigcup_{1 \leq i \leq j} \Omega_i} A_\alpha$. As for any a

from Q_1 $\text{less}(a, Q_3)$ belongs to $\bigcup_{i=1}^\infty \Omega'_i$, then for any a from Q_1 we have the number $\text{less}(a, Q_2)$. So, we get such numbers :

- $\text{less}(a, Q_2) \forall a \in Q_1;$
- $\text{less}(a, Q_3) \forall a \in Q_1;$
- $\text{less}(b, Q_3) \forall b \in Q_2.$

Now we show that one can obtain the elements of the blocks Q_1, Q_2, Q_3 from these numbers. Really, we can get all the numbers of the first block. Indeed, for some $a_j \in Q_1$ we have:

$$a_j = \text{less}(a_j, Q_3) + \text{less}(a_j, Q_2) + 1 + j.$$

Next, for $b_j \in Q_2$ we have that $\text{less}(b_j, Q_3) = |\{a \in Q_1 : \text{less}(a, Q_2) < j\}|$; and for $c_j \in Q_3$ it is true that $\text{less}(c_j, Q_2) = |\{b \in Q_2 : \text{less}(b, Q_3) < j\}|$ and $\text{less}(c_j, Q_1) = |\{b \in Q_1 : \text{less}(b, Q_3) < j\}|$.

Hence we get elements of blocks Q_2 and Q_3 :

$$b_j = \text{less}(b_j, Q_3) + \text{less}(b_j, Q_1) + j + 1;$$

$$c_j = \text{less}(c_j, Q_1) + \text{less}(c_j, Q_2) + j + 1.$$

So, for abstract properties of semigroup \mathcal{T} it is possible to restore the corresponding partition from $\Lambda(\mathbb{N}, 3)$; then it means that non-isomorphic semigroups correspond to different partitions, so the theorem is proved for $k = 3$.

Now suppose the statement of the theorem holds for all $k \leq k_0$. Let \mathcal{T} be a semigroup from $Nil(\mathbb{N}, k_0 + 1)$, and partition λ is the respective partition from $\Lambda(\mathbb{N}, k_0 + 1)$ with blocks $Q_1, Q_2, \dots, Q_{k_0}, Q_{k_0+1}$. Now let's consider the set

$$S_1 = \{s \in \mathcal{T} : \forall a_1, \dots, a_{k_0-1} \in \mathcal{T} \quad s \cdot a_1 \cdot \dots \cdot a_{k_0-1} = 0\}.$$

It is easy to see that a transformation of \mathcal{T} belongs to S_1 if and only if its range has empty intersection with the second block of the partition λ . It is also obvious that S_1 is a subsemigroup of \mathcal{T} . More, S_1 belongs to $Nil(\mathcal{T}, k_0)$. Really, there is a correspondent partition from $\Lambda(\mathcal{T}, k_0)$ with blocks $Q_1 \cup Q_2, \dots, Q_{k_0+1}$. Then for the induction assumption one can obtain the numbers of blocks $Q_1 \cup Q_2, Q_3, \dots, Q_{k_0+1}$. Next, let's consider the set

$$S_2 = \{s \in \mathcal{T} : \forall a_1, \dots, a_{k_0-1} \in \mathcal{T} \quad a_1 \cdot \dots \cdot a_{k_0-1} \cdot s = 0\}.$$

Analogously, a transformation of \mathcal{T} belongs to S_2 if and only if its domain has empty intersection with the next to the last block of the partition λ ; and S_2 is a maximal nilpotent subsemigroup of nilpotency class k_0 with a corresponding partition having blocks $Q_1, Q_2, \dots, Q_{k_0} \cup Q_{k_0+1}$. For the induction assumption one can obtain the numbers of the first block Q_1 . So, we have the numbers of all the blocks of the partition λ . So, for the properties \mathcal{T} as abstract semigroup we get elements of the blocks of the correspondent partition, so the theorem is proved. \square

Corollary 2. *Let $Nil(n, k)$ denote the set of all maximal nilpotent sub-semigroups of the semigroup of order-decreasing transformations of the set $\{1, \dots, n\}$. Then all the semigroups from $\bigcup_{\substack{n \geq 4 \\ 3 \leq k \leq n-1}} Nil(n, k)$ are pairwise*

non-isomorphic.

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