Algebra and Discrete Mathematics
Number 4. **(2006).** pp. 67 – 80

RESEARCH ARTICLE

Dense ideal extensions of strict regular semigroups

F. J. Pastijn and L. Oliveira

Communicated by L. Marki

Abstract. If $\mathbf{V}$ is an existence variety of strict regular semigroups, then every semigroup in $\mathbf{V}$ has, within $\mathbf{V}$, a maximal dense ideal extension.

1. Introduction

For a general background on semigroup theory we refer to [4], [8], [9], [18], [19]. We shall assume that the reader has some familiarity with pseudosemilattices [13]. It will be useful to review some basic results first.

Result 1.1. Let S be a regular semigroup and $a, b \in S$ such that $J_a \leq J_b$. Then there exists $c \in D_a$ such that $c \leq b$.

Proof. There exist $s, t \in S^1$ such that $a = sbt$. Let e and f be idempotents of S such that $e \,\mathcal{R}\, bt$ and $f \,\mathcal{L}\, seb$. Then $a = sbt \,\mathcal{R}\, seb \,\mathcal{L}\, ebf \leq eb \leq b$. □

A regular semigroup S is called **completely semisimple** if every principal factor of S is completely simple or completely 0-simple [12]. Applying Result 1.1 twice we easily prove that if S is a regular semigroup and $a \in S$ such that $D_a \neq J_a$, then D_a contains distinct comparable idempotents. >From this observation it follows that a regular semigroup S is completely semisimple if and only if no $\mathcal{D}$-class of S contains distinct

The second author was partially supported by FCT through Centro de Matemática da Universidade do Porto

comparable idempotents. It should also be clear now that for a completely semisimple semigroup the Green relations $\mathcal{J}$ and $\mathcal{D}$ coincide.

In view of Result 1.1 it is now natural to introduce the following. A regular semigroup S is said to be **strict** if for all $a, b \in S$ such that $J_a \leq J_b$ there exists a unique $c \in D_a$ such that $c \leq b$. From what we have just seen it follows that every strict regular semigroup S is completely semisimple and in particular, $\mathcal{J} = \mathcal{D}$ in S. Strict regular semigroups have been investigated thoroughly in the past four decades and the present paper is one more contribution. We shall explain below what we intend to do.

For a regular semigroup S in general there is a natural way to introduce a quasi-order on $S/\mathcal{D}$ and a partial order on S/J. When dealing with a strict regular semigroup S, we have that $\mathcal{J} = \mathcal{D}$ for S and we shall have no qualms referring to the poset $I = S/\mathcal{D}$. In this circumstance, if D_α is the $\mathcal{D}$-class corresponding to $\alpha \in I$, then we refer to $(D_\alpha,\ \alpha \in I)$ as the poset of $\mathcal{D}$-classes of S. If D_α is a $\mathcal{D}$-class of the strict regular semigroup S, let D_α^0 be the set D_α with an extra 0 adjoined. If α is not the least element of $I = S/\mathcal{D}$ then we interpret D_α^0 as a principal factor of S and otherwise D_α^0 is the completely simple semigroup D_α with a zero adjoined. Then the direct product $\prod_{\alpha \in I} D_\alpha^0$ has a largest maximal ideal given by

$$M = \{(a_\alpha,\ \alpha \in I) \mid a_\alpha = 0 \ \text{for some} \ \alpha \in I\}.$$

The Rees quotient $(\prod_{\alpha \in I} D_\alpha^0)/M$ will be denoted by $\prod^0_{\alpha \in I} D_\alpha^0$ and will be called the 0-**direct product** of the D_α^0, $\alpha \in I$. One verifies that $\prod^0_{\alpha \in I} D_\alpha^0$ is a completely 0-simple semigroup.

The following key result finds its origin in [11].

Result 1.2. (*i*) Every strict regular semigroup is isomorphic to a subdirect product of its principal factors.

(*ii*) A regular semigroup is strict if and only if it is isomorphic to a subdirect product of completely simple and/or completely 0-simple semigroups.

A class of regular semigroups is called an **existence variety** if it is closed for the taking of homomorphic images, regular subsemigroups and direct products. The study of existence varieties was initiated in [5] and [10]. A regular semigroup S is said to be **locally inverse** [**locally a Clifford semigroup**] if for every idempotent e of S, eSe is an inverse [Clifford] semigroup. The class **SR** of all strict regular semigroups is contained in the class **LI** of all locally inverse semigroups, and both classes are existence varieties [5]. In fact, as observed in [5]

Result 1.3. A regular semigroup is strict if and only if it is locally a Clifford semigroup.

Naturally, if S is a regular semigroup, then the existence variety **generated** by S is the smallest existence variety containing S.

Proposition 1.4. An existence variety consists of strict regular semigroups only, if and only if it is generated by a completely simple or by a completely 0-simple semigroup.

Proof. Every existence variety $\mathbf{V}$ of locally inverse semigroups is generated by a single locally inverse semigroup, namely the so-called bifree object in $\mathbf{V}$ on a countably infinite set of variables ([1], [24]). If $\mathbf{V}$ consists of completely simple semigroups only then this bifree object is completely simple, and $\mathbf{V}$ is generated by this completely simple semigroup.

We shall from now on assume that $\mathbf{V}$ is an existence variety consisting of strict regular semigroups only, but that not all members of $\mathbf{V}$ are completely simple. Let S be a strict regular semigroup which generates $\mathbf{V}$. Then S is not completely simple and, if $(D_\alpha,\ \alpha \in I)$ is the poset of $\mathcal{D}$-classes of S, then by Result 1.2 the 0-direct product $\prod^0_{\alpha\in I} D^0_\alpha$ generates $\mathbf{V}$. This semigroup $\prod^0_{\alpha\in I} D^0_\alpha$ is completely 0-simple.

The converse part is obvious. □

As an example, the existence variety consisting of the strict regular semigroups S for which eSe is a semilattice for every idempotent e of S (see [13], [14]) is generated by a 5-element completely 0-simple semigroup (see [5], [6]) and there are precisely 12 existence varieties properly contained in this one. This existence variety is precisely the class of all strict regular semigroups that are also combinatorial. For an account of the lattice of existence varieties each consisting of strict inverse semigroups, we refer to Section XII.4 of [19]. An existence variety consists of normal bands of groups only, if and only if it is generated by a completely simple semigroup or by a completely simple semigroup with a zero adjoined. We refer to [21] for further details about the lattice of existence varieties consisting of strict regular semigroups. Since this lattice is of the power of the continuum we retain from Proposition 1.4 that the concept of existence varieties is a powerful tool for classifying completely 0-simple semigroups.

If $\mathbf{V}$ is an existence variety, $S, T \in \mathbf{V}$ and $\varphi : S \longrightarrow T$ an injective homomorphism such that $S\varphi$ is an ideal of T, then we say that $\varphi : S \longrightarrow T$ is an ideal extension of S within $\mathbf{V}$. An ideal extension $\varphi : S \longrightarrow T$ within $\mathbf{V}$ is called dense if whenever $\psi : T \longrightarrow U$ is a homomorphism such that $\varphi\psi : S \longrightarrow U$ is an ideal extension within $\mathbf{V}$, then ψ is injective.

A dense ideal extension $\varphi : S \longrightarrow T$ within $\mathbf{V}$ is called maximal if whenever $\psi : T \longrightarrow U$ is a homomorphism such that $\varphi\psi : S \longrightarrow U$ is a dense ideal extension within $\mathbf{V}$, then ψ is an isomorphism of T onto U. The construction of ideal extensions within $\mathbf{V}$ simplifies considerably if for every $S \in \mathbf{V}$, there exists a maximal dense ideal extension of S within $\mathbf{V}$. We refer to [15], [16], [17], [18] for a background on ideal extensions and dense ideal extensions, in general, and within the existence variety of locally inverse semigroups in particular. The main goal of this paper will be to show that there always exist maximal dense ideal extensions within every existence variety consisting of strict regular semigroups.

For any regular semigroup S the set of order ideals of S (with respect to the natural partial order) forms a subsemigroup $O(S)$ of the power semigroup on S, and the mapping $\tau_S : S \longrightarrow O(S)$ which associates with $a \in S$ the principal order ideal $(a]$ is a faithful representation of S if and only if S is locally inverse [15]. If $T(S)$ denotes the regular part of the idealizer of $S\tau_S$ in $O(S)$, then $\tau_S : S \longrightarrow T(S)$ is a maximal dense ideal extension of S within $\mathbf{LI}$ [15]. From [20] it follows that if S is a normal band of groups, then $\tau_S : S \longrightarrow T(S)$ is a maximal dense ideal extension of S within the existence variety generated by S. In [20] Petrich uses threads to great advantage, and we shall follow suit in Section 2. Our main result features in Section 3.

2. Strict regular semigroups of threads

Let S be a locally inverse semigroup and $O(S)$ the semigroup of order ideals of S. For $H, H' \in O(S)$ we shall say that H and H' are **pairwise inverse threads** of S if $H = HH'H$, $H' = H'HH'$, and, HH' and $H'H$ are subsemilattices of S. Further, $H \in O(S)$ is called a **thread** of S if for some $H' \in O(S)$, H and H' are pairwise inverse threads. The set of all threads of S is denoted by $C(S)$. Threads were used for inverse semigroups in [23], for normal bands of groups in [20] and for locally inverse semigroups in general in [17].

Given a locally inverse semigroup S, $C(S)$ need not constitute a subsemigroup of $O(S)$, even in the case where S is a completely 0-simple semigroup (see Corollary 2.7 of [17]). But $C(S)$ forms a subsemigroup of $O(S)$ if S is an inverse semigroup [23], or a normal band of groups [20] or a straight locally inverse semigroup [17]. For a locally inverse semigroup S in general we have $S\tau_S \subseteq T(S) \subseteq C(S) \subseteq O(S)$ [15], [17].

If $S \in \mathbf{SR}$ is a strict regular semigroup, then we let $C_{\mathbf{SR}}(S)$ consist of the $H \in C(S)$ such that H intersects every $\mathcal{D}$-class of S in at most one element. If $a \in S$, then we have from the definition of $\mathbf{SR}$ that $(a] \in C_{\mathbf{SR}}(S)$ and therefore $S\tau_S \subseteq C_{\mathbf{SR}}(S) \subseteq C(S) \subseteq O(S)$. The remaining theorem of this section states that $C_{\mathbf{SR}}(S)$ is the largest strict regular subsemigroup of $O(S)$ which is contained in $C(S)$ and contains $S\tau_S$.

We collect some auxiliary results from [15] and [17].

Result 2.1 (Lemmas 3.2 and 3.11 of [15], Result 2.1 of [17])**.** Let S be a locally inverse semigroup.

(*i*) If $H \in C(S)$ then no distinct elements of H are $\mathcal{L}$- or $\mathcal{R}$-related in S.

(*ii*) For $F \in C(S)$, F is an idempotent of $O(S)$ if and only if it is a subsemilattice of S.

(*iii*) Let H and H' be pairwise inverse threads and $E = HH'$. Then for every $a \in H$ there exists a unique inverse a' of a which belongs to H' and the mapping $H \longrightarrow H'$, $a \longrightarrow a'$ is an order isomorphism. For every $e \in E$ there exist unique $h \in H$ and $h' \in H'$ such that $h\,\mathcal{R}\,e\,\mathcal{L}\,h'$ and then h and h' are pairwise inverse elements of S with $e = hh'$.

Result 2.2 (Lemma 3.3 of [15])**.** Let S be a locally inverse semigroup and $E, F \in C(S)$ idempotents of $O(S)$. Put $G = \{e \wedge f \mid e \in E,\ f \in F\}$. Then

(*i*) $G \in O(S)$ and G is a subpseudosemilattice of $(E(S), \wedge)$,

(*ii*) $G = \{g \in E(S) \mid e\,\mathcal{R}\,g\,\mathcal{L}\,f$ for some $e \in E,\ f \in F\}$,

(*iii*) $EG = G = GF$.

Lemma 2.3. Let S be a locally inverse semigroup and (H, H'), (K, K') pairs of pairwise inverse threads of S. Let $E = KK'$, $F = H'H$ and $G = \{e \wedge f \mid e \in E,\ f \in F\}$. If G is an idempotent of $C(S)$, then $GK \subseteq K$, $HG \subseteq H$, $HGK = HK$, and, HK and $K'GH'$ are pairwise inverse threads of S.

Proof. We assume that G is an idempotent of $C(S)$ and thus by Result 2.1, G is a subsemilattice and an order ideal of S.

Let $gb \in GK$ for some $g \in G$ and $b \in K$. By Result 2.1 there exists a unique inverse b' of b in K'. We let $h = bb' \wedge g$ and see that $gb = ghb$. By Result 2.2 there exists $f \in F$ such that $g\,\mathcal{L}\,f$, thus $h = bb' \wedge f$. Therefore $h \in G$ since $bb' \in E$. Since $h = hg$ in the semilattice G it follows that $h \leq g$ and therefore also that $gb = ghb = hb \leq b$, whence $gb \in K$ since K is an order ideal. We proved that $GK \subseteq K$, and in a dual way we can prove that $HG \subseteq H$.

>From the foregoing we have that $K'GK \subseteq K'K$ and so $K'GK$ is a subsemilattice and an order ideal of S by Result 2.1. We show in a dual way that the same is true for HGH'. Furthermore by Result 2.2, $KK'G = G = GH'H$. Thus

$$(HGK)(K'GH') = HGH',$$
$$(K'GH')(HGK) = K'GK,$$
$$(HGK)(K'GH')(HGK) = HGK,$$
$$(K'GH')(HGK)(K'GH') = K'GH',$$

and we see that HGK and $K'GH'$ are pairwise inverse threads of S.

It remains to show that $HK = HGK$. From $HG \subseteq H$ we already have that $HGK \subseteq HK$. Let $a \in H$, $b \in K$ and, using Result 2.1, let a' and b' be the inverses of a and b in H' and K', respectively. Put $f = a'a$, $e = bb'$, thus $e \wedge f \in G$. Then $ab = a(e \wedge f)b \in HGK$ and we may conclude that $HK = HGK$. □

>From here on we confine ourselves to strict regular semigroups.

Lemma 2.4. Let S be a strict regular semigroup. For $H \in C(S)$ the following are equivalent:

(*i*) $H \in C_{\mathbf{SR}}(S)$,

(*ii*) if (H, H') is some [any] pair of pairwise inverse threads of S, then $H' \in C_{\mathbf{SR}}(S)$,

(*iii*) if (H, H') is some [any] pair of pairwise inverse threads of S, then $HH' \in C_{\mathbf{SR}}(S)$.

Proof. Assume that $H \in C_{\mathbf{SR}}(S)$ and (H, H') is any pair of pairwise inverse threads of S. Let D be a $\mathcal{D}$-class of S and $c, d \in H' \cap D$. By Result 2.1 there exist $a, b \in H$ such that a and b are inverses of c and d, respectively. Then $a, b \in H \cap D$ and since H intersects D in at most one element we need to have $a = b$. Therefore $ac \,\mathcal{R}\, ad$ and $ca \,\mathcal{L}\, da$, and since HH' and $H'H$ are semilattices, we need to have that $ac = ad$ and $ca = da$. Therefore $c \,\mathcal{H}\, d$ and thus $c = d$ by Result 2.1. We conclude that $H' \in C_{\mathbf{SR}}(S)$. We proved that (*i*) and (*ii*) are equivalent statements.

Assume that $H \in C_{\mathbf{SR}}(S)$ and let $H' \in C(S)$ such that H and H' are pairwise inverse threads. Then HH' is a subsemilattice and an order ideal of S by Result 2.1 and $H' \in C_{\mathbf{SR}}(S)$ by the above. Let D be a $\mathcal{D}$-class of S and $e, f \in HH' \cap D$. By Result 2.1 there exist $a, b \in H$, $a', b' \in H'$ such that a' is an inverse of a, b' is an inverse of b, $a \,\mathcal{R}\, e \,\mathcal{L}\, a'$,

$b\,\mathcal{R}\,f\,\mathcal{L}\,b'$ and $aa' = e$, $bb' = f$. Since $e\,\mathcal{D}\,f$, so also $a\,\mathcal{D}\,b$ and $a'\,\mathcal{D}\,b'$, thus $a = b$ and $a' = b'$ since $H, H' \in C_{\mathbf{SR}}(S)$. Therefore $e = f$. We proved that the equivalent conditions (i) and (ii) each imply (iii).

Assume that H and H' are pairwise inverse threads of S such that $HH' \in C_{\mathbf{SR}}(S)$. Let $a, b \in H$ such that $a\,\mathcal{D}\,b$. By Result 2.1 there exist $a', b' \in H'$ such that a' and b' are inverses of a and b respectively. Then $aa'\,\mathcal{R}\,a\,\mathcal{D}\,b\,\mathcal{R}\,bb'$ with $aa', bb' \in HH'$, whence $aa' = bb'$ and so $a\,\mathcal{R}\,b$. By Result 2.1 it follows that $a = b$. Therefore $H \in C_{\mathbf{SR}}(S)$. We proved that (iii) implies (i). □

Theorem 2.5. If S is a strict regular semigroup, then $C_{\mathbf{SR}}(S)$ is the largest strict regular subsemigroup of $O(S)$ which is contained in $C(S)$ and contains $S\tau_S$.

Proof. For the strict regular semigroup S we let E and F be idempotents of $C_{\mathbf{SR}}(S)$. In particular, by Result 2.1, E and F are both subsemilattices and order ideals of S. We put $G = \{e \wedge f \mid e \in E,\ f \in F\}$. Assume that $g_1, g_2 \in G$ such that $g_1\,\mathcal{D}\,g_2$ in S. By Result 2.2 there exist $e_1, e_2 \in E$ and $f_1, f_2 \in F$ such that $e_1\,\mathcal{R}\,g_1\,\mathcal{L}\,f_1$ and $e_2\,\mathcal{R}\,g_2\,\mathcal{L}\,f_2$. Then $e_1\,\mathcal{D}\,e_2$ and $f_1\,\mathcal{D}\,f_2$, hence $e_1 = e_2$, $f_1 = f_2$ since $E, F \in C_{\mathbf{SR}}(S)$, and so $g_1 = g_2$. By Result 2.2, G is an order ideal of S which is also a subpseudosemilattice of $(E(S), \wedge)$ and by the foregoing no two distinct elements of G can be $\mathcal{L}$- or $\mathcal{R}$-related. It follows that G is a subsemilattice and an order ideal of S, and thus an idempotent of $C(S)$, by Result 2.1. From the above it follows that moreover, G intersects every $\mathcal{D}$-class of S in at most one element and so we conclude that G is an idempotent of $C_{\mathbf{SR}}(S)$.

We now let $H, K \in C_{\mathbf{SR}}(S)$ and we let (H, H'), (K, K') be pairs of pairwise inverse threads of S. We put $E = KK'$, $F = H'H$ and $G = \{e \wedge f \mid e \in E,\ f \in F\}$. By Lemma 2.4 and the above, $H, H', K, K', E, F, G \in C_{\mathbf{SR}}(S)$. By Lemma 2.3, HK and $K'GH'$ are pairwise inverse threads of S, and using Result 2.2 and Lemma 2.3, $(HK)(K'GH') \subseteq HH'$, where HH' is an idempotent of $C_{\mathbf{SR}}(S)$ by Lemma 2.4. It follows that $(HK)(K'GH')$ is an idempotent of $C_{\mathbf{SR}}(S)$ and so again by Lemma 2.4, HK and $K'GH'$ are pairwise inverse threads which belong to $C_{\mathbf{SR}}(S)$. It follows that $C_{\mathbf{SR}}(S)$ is a regular subsemigroup of $O(S)$ which is contained in $C(S)$. By Proposition 2.4 of [16], $C_{\mathbf{SR}}(S)$ is a locally inverse semigroup.

Let E be an idempotent of $C_{\mathbf{SR}}(S)$ and H, H' pairwise inverse elements in the inverse semigroup $E\,C_{\mathbf{SR}}(S)\,E$. In particular H and H' are pairwise inverse threads. We put $F = HH'$ and $G = H'H$. In view of Result 1.3 we need to show that $E\,C_{\mathbf{SR}}(S)\,E$ is a Clifford semigroup. In order to show that $E\,C_{\mathbf{SR}}(S)\,E$ is a Clifford semigroup, it now suffices to show that $F = G$. It follows from Proposition 2.3 of [17] that the semilattices F and G are order ideals of the semilattice E.

For $f \in F$, $f = hh'$ for some pairwise inverse elements $h \in H$ and $h' \in H'$ by Result 2.1. Putting $g = h'h \in H'H = G$ we thus have $f \mathcal{D} g$ in S. Since $f, g \in E$ and $E \in C_{\mathbf{SR}}(S)$ it follows that $f = g$. Using symmetry we conclude that $F = G$. Therefore $E\, C_{\mathbf{SR}}(S)\, E$ is a Clifford semigroup, and we conclude that $C_{\mathbf{SR}}(S)$ is a strict regular semigroup.

Let C be a strict regular subsemigroup of $O(S)$ which is contained in $C(S)$ and contains $S\tau_S$. Let E be an idempotent of C. By Result 2.1, E is a subsemilattice and an order ideal of S. Let $e, f \in E$ and $e \mathcal{D} f$ in S. Then $(e]$ and $(f]$ are $\mathcal{D}$-related idempotents of $S\tau_S$ and thus also of C, and $(e] \leq E$, $(f] \leq E$ by Proposition 2.3 of [17]. It follows that $(e] = (f]$ and thus $e = f$. We conclude that $E \in C_{\mathbf{SR}}(S)$. By Lemma 2.4, $C \subseteq C_{\mathbf{SR}}(S)$. □

3. Dense ideal extensions within SR

For any strict regular semigroup S we define $T_{\mathbf{SR}}(S) = T(S) \cap C_{\mathbf{SR}}(S)$. Then clearly $S\tau_S \subseteq T_{\mathbf{SR}}(S)$ and so $\tau_S : S \longrightarrow T_{\mathbf{SR}}(S)$ is an embedding. In this section we shall show that this embedding gives rise to a maximal dense ideal extension of S, not only within **SR**, but in fact also within the existence variety generated by S.

Theorem 3.1. Let S be a strict regular semigroup and T a regular semigroup such that $S\tau_S \subseteq T \subseteq T_{\mathbf{SR}}(S)$. Then $\tau_S : S \longrightarrow T$ is a dense ideal extension of S within **SR**. Conversely, every dense ideal extension of S within **SR** is equivalent to a unique ideal extension $\tau_S : S \longrightarrow T$ obtained in this way.

Proof. Let T be a regular semigroup such that $S\tau_S \subseteq T \subseteq T_{\mathbf{SR}}(S)$. Then by Theorem 2.5, T is a strict regular semigroup and in view of Theorem 4.6 of [15], $\tau_S : S \longrightarrow T$ is a dense ideal extension within **SR**.

Again by Theorem 4.6 of [15] every dense ideal extension of S within **SR** is equivalent to a unique dense ideal extension $\tau_S : S \longrightarrow T$ where $T \subseteq T(S)$. Here of course T itself must be a strict regular semigroup, and $T \subseteq C(S) \subseteq O(S)$ [15], [17]. Therefore by Theorem 2.5, $T \subseteq C_{\mathbf{SR}}(S)$. Hence $T \subseteq T_{\mathbf{SR}}(S)$. □

Theorem 3.2. Let S be a strict regular semigroup. Then $\tau_S : S \longrightarrow T_{\mathbf{SR}}(S)$ is a maximal dense ideal extension within **SR**.

Proof. Let $H \in T_{\mathbf{SR}}(S)$ and $H' \in T(S)$ an inverse of H in $T(S)$. Then $H' \in C(S)$ is an inverse of H in $C_{\mathbf{SR}}(S)$ by Lemma 2.4, whence H and H'

are pairwise inverse elements in $T_{\mathbf{SR}}(S)$. It follows that $T_{\mathbf{SR}}(S)$ is a regular semigroup. Further, since $T_{\mathbf{SR}}(S) \subseteq C_{\mathbf{SR}}(S)$ we have from Theorem 2.5 that $T_{\mathbf{SR}}(S)$ is a strict regular semigroup. Therefore by Theorem 3.1, $\tau_S : S \longrightarrow T_{\mathbf{SR}}(S)$ is a dense ideal extension within **SR**. We proceed to show that this ideal extension is a maximal dense ideal extension within **SR**.

Let $\psi : T_{\mathbf{SR}}(S) \longrightarrow U$ be a homomorphism such that $\tau_S\psi : S \longrightarrow U$ is a dense ideal extension within **SR**. Then U is a strict regular semigroup. We need to show that ψ is an isomorphism. By Theorem 3.1 there exists a unique strict regular semigroup T such that $S\tau_S \subseteq T \subseteq T_{\mathbf{SR}}(S)$ such that the dense ideal extensions $\tau_S : S \longrightarrow T$ and $\tau_S\psi : S \longrightarrow U$ are equivalent. By Corollary 4.5 of [15],

$$\zeta : U \longrightarrow T(S), \quad u \longrightarrow ((u] \cap S\tau_S\psi)(\tau_S\psi)^{-1}$$

is the unique homomorphism of U into $T(S)$ which extends $(\tau_S\psi)^{-1}\tau_S$, and ζ is injective since $\tau_S\psi : S \longrightarrow U$ is a dense ideal extension. Clearly then, the ideal extensions $\tau_S : S \longrightarrow U\zeta$ and $\tau_S\psi : S \longrightarrow U$ are equivalent, whence $T = U\zeta$.

It remains to show that $T = T_{\mathbf{SR}}(S)$ and that ψ and ζ are pairwise inverse isomorphisms. The proof for this follows the same lines as in the proof of Theorem 4.7 of [15]. □

We shall need the following result from [17].

Result 3.3 (Proposition 2.3 of [17])**.** Let S be a locally inverse semigroup and T a regular subsemigroup of $O(S)$ such that $T \subseteq C(S)$. Then

(*i*) $H \,\mathcal{R}\, K$ in T if and only if there exists a bijection $\varphi : H \longrightarrow K$ such that $h \,\mathcal{R}\, h\varphi$ for every $h \in H$,

(*ii*) $H \,\mathcal{L}\, K$ in T if and only if there exists a bijection $\varphi : H \longrightarrow K$ such that $h \,\mathcal{L}\, h\varphi$ for every $h \in H$.

With the notation of the introduction we have

Lemma 3.4. Let S be a strict regular semigroup and $(D_\alpha,\ \alpha \in I)$ the poset of $\mathcal{D}$-classes of S. Then every principal factor of $C_{\mathbf{SR}}(S)$ $[T_{\mathbf{SR}}(S)]$ can be isomorphically embedded into the completely 0-simple semigroup $\prod^0_{\alpha\in I} D^0_\alpha$.

Proof. >From Theorems 2.5 and 3.2 we know that $C_{\mathbf{SR}}(S)$ and $T_{\mathbf{SR}}(S)$ are strict regular semigroups and thus in particular, each principal factor of $C_{\mathbf{SR}}(S)$ and of $T_{\mathbf{SR}}(S)$ is a completely simple or completely 0-simple semigroup. Since $T_{\mathbf{SR}}(S)$ is a regular subsemigroup of $C_{\mathbf{SR}}(S)$, each principal factor of $T_{\mathbf{SR}}(S)$ is a subsemigroup of a principal factor of $C_{\mathbf{SR}}(S)$. Thus it suffices to prove the statement of the lemma for $C_{\mathbf{SR}}(S)$ only.

Let D be a $\mathcal{D}$-class of $C_{\mathbf{SR}}(S)$ and let $H, K \in D$. By Result 3.3 and Theorem 2.5 it follows that

$$\{\alpha \in I \mid D_\alpha \cap H \neq \emptyset\} = \{\alpha \in I \mid D_\alpha \cap K \neq \emptyset\}.$$

This justifies the notation

$$I_D = \{\alpha \in I \mid D_\alpha \cap H \neq \emptyset\}$$

where H is some [any] element of D. By Result 1.1, I_D is an order ideal of I.

We have that $|I_D| = 1$ if and only if D consists of the $(a]$, where $a \in D_\gamma$, with γ being the least element of I. If this is the case then $D \cong D_\gamma$ is completely simple and D can be isomorphically embedded into $\prod^0_{\alpha \in I} D^0_\alpha$.

We shall henceforth assume that $|I_D| \geq 2$. Then D cannot be the least element in the poset of $\mathcal{D}$-classes of $C_{\mathbf{SR}}(S)$ and so the completely 0-simple semigroup D^0 is a principal factor of $C_{\mathbf{SR}}(S)$. We consider the mapping

$$\varphi : D^0 \longrightarrow \textstyle\prod^0_{\alpha \in I_D} D^0_\alpha, \qquad H \longrightarrow (a_\alpha,\ \alpha \in I_D),\ H \in D$$
$$0 \longrightarrow 0,$$

where for every $\alpha \in I_D$, $D_\alpha \cap H = \{a_\alpha\}$. The mapping φ is obviously injective. We need to show that φ is a homomorphism. Since both D^0 and $\prod^0_{\alpha \in I_D} D^0_\alpha$ are completely 0-simple it suffices to show that for $H, K \in D$, $HK \neq 0$ if and only if $(H\varphi)(K\varphi) \neq 0$, and if this is the case, then $(HK)\varphi = (H\varphi)(K\varphi)$.

In the following we shall choose $H, K \in D$ and we put $H\varphi = (a_\alpha\,,\ \alpha \in I_D)$ and $K\varphi = (b_\alpha\,,\ \alpha \in I_D)$. Assume that $HK \neq 0$ in the completely 0-simple semigroup D^0. There exists an idempotent $G \in D$ such that $H\,\mathcal{L}\,G\,\mathcal{R}\,K$. By Result 2.1, G is a subsemilattice and an order ideal of S. We put $G\varphi = (g_\alpha, \alpha \in I_D)$, and using Result 3.3 we have that $a_\alpha\,\mathcal{L}\,g_\alpha\,\mathcal{R}\,b_\alpha$ for every $\alpha \in I_D$. It follows that $(H\varphi)(K\varphi) = (a_\alpha b_\alpha, \alpha \in I_D) \neq 0$, where for every $\alpha \in I_D$, $a_\alpha b_\alpha \in D_\alpha \cap HK$. From this it follows that $(HK)\varphi = (H\varphi)(K\varphi)$.

Assume that conversely $(H\varphi)(K\varphi) \neq 0$. Then $a_\alpha b_\alpha \neq 0$ for every $\alpha \in I_D$, and so there exists an idempotent $g_\alpha \in D_\alpha$ such that $a_\alpha\,\mathcal{L}\,g_\alpha\,\mathcal{R}\,b_\alpha$ for every $\alpha \in I_D$. We put $G = \{g_\alpha \mid \alpha \in I_D\}$. Let E and F be idempotents of D such that $E\,\mathcal{L}\,H$ and $F\,\mathcal{R}\,K$ in D. Then $E\varphi = (e_\alpha, \alpha \in I_D)$,

$F\varphi = (f_\alpha,\, \alpha \in I_D)$ and by Result 3.3, $e_\alpha \,\mathcal{L}\, g_\alpha \,\mathcal{R}\, f_\alpha$ for every $\alpha \in I_D$. By Result 2.2, $G = \{f \wedge e \,|\, e \in E,\, f \in F\}$, and by Result 3.3, G is an idempotent of $C_{\mathbf{SR}}(S)$. Again by Result 3.3, $H \,\mathcal{L}\, G \,\mathcal{R}\, K$ in D, whence $HK \neq 0$ in D^0.

It should be obvious that $\prod^0_{\alpha \in I_D} D^0_\alpha$ can be embedded into $\prod^0_{\alpha \in I} D^0_\alpha$. Hence the required result follows. □

Corollary 3.5. If S is a strict regular semigroup, then the existence varieties each generated by S, $T_{\mathbf{SR}}(S)$ and $C_{\mathbf{SR}}(S)$ coincide.

Proof. If S is a completely simple semigroup then $S \cong C_{\mathbf{SR}}(S) \cong T_{\mathbf{SR}}(S)$ and the statement is obvious. Otherwise, as in the proof of Proposition 1.4 and with the notation of Lemma 3.4, S and $\prod^0_{\alpha \in I} D^0_\alpha$ generate the same existence variety. Since $C_{\mathbf{SR}}(S)$ is a strict regular semigroup which is not completely simple, the existence variety generated by $C_{\mathbf{SR}}(S)$ is contained in the existence variety generated by $\prod^0_{\alpha \in I} D^0_\alpha$ by Lemma 3.4. Since $S \cong S\tau_S \subseteq T_{\mathbf{SR}}(S) \subseteq C_{\mathbf{SR}}(S)$ we may now conclude that the existence varieties each generated by S, $T_{\mathbf{SR}}(S)$ and $C_{\mathbf{SR}}(S)$ coincide. □

Theorem 3.6. Let $\mathbf{V}$ be an existence variety of strict regular semigroups. Then for every $S \in \mathbf{V}$, $\tau_S : S \longrightarrow T_{\mathbf{SR}}(S)$ is a maximal dense ideal extension within $\mathbf{V}$.

Proof. The statement follows from Theorem 3.2 and Corollary 3.5. □

4. Problems

>From what we have seen, the following question comes naturally: characterize the existence varieties $\mathbf{V}$ of locally inverse semigroups which satisfy the property that for every $S \in \mathbf{V}$ has within $\mathbf{V}$ a maximal dense ideal extension. At this time we only know of the following existence varieties which satisfy this pleasant property: (i) the existence variety $\mathbf{LI}$ of locally inverse semigroups [15], (ii) the (existence) variety $\mathbf{I}$ of all inverse semigroups [22] and (iii) every existence variety contained in $\mathbf{SR}$. There are undoubtedly more existence varieties satisfying our request, and our general question may well turn out to be a tool for locating interesting existence varieties.

For a locally inverse semigroup S, the ideal extension $\tau_S : S \longrightarrow T(S)$ is a maximal dense ideal extension of S within $\mathbf{LI}$ [15]. Another natural question to ask is which existence varieties $\mathbf{V}$ of locally inverse semigroup satisfy the stronger condition:

$$S \in \mathbf{V} \implies T(S) \in \mathbf{V}\,. \qquad (4.1)$$

If S is a normal band of groups, then we immediately conclude that $C(S) = C_{\mathbf{SR}}(S)$, and therefore $T(S) = T_{\mathbf{SR}}(S)$. It now follows from Theorem 3.6 that every existence variety of normal bands of groups satisfies this condition.

Every existence variety $\mathbf{V}$ satisfying condition (4.1) and containing the 5-element combinatorial Brandt semigroup must contain $\mathbf{I}$ (see e.g. [19]). Recall that the existence variety $\mathbf{NBG}$ of all normal bands of groups is the largest existence variety of locally inverse semigroups not containing the 5-element combinatorial Brandt semigroup [5]. Let A_2 be the 5-element completely 0-simple semigroup which generates the existence variety of all combinatorial strict regular semigroups. In a future paper we shall show that $\mathbf{LI}$ is the only existence variety of locally inverse semigroups with property (4.1) and containing A_2. Any existence variety not containing A_2 must be an existence variety of E-solid regular semigroups [6], that is, an existence variety of regular semigroups whose idempotents generate a completely regular semigroup. Thus, any other existence variety of locally inverse semigroups with property (4.1) must be an existence variety of E-solid locally inverse semigroups.

Let $\mathbf{ESLI}$ be the class of all E-solid locally inverse semigroups. Thus $\mathbf{ESLI}$ consists of all regular semigroups for which the idempotents generate a normal band of groups. We shall denote by $\mathbf{GI}$ the subclass of $\mathbf{ESLI}$ of all generalized inverse semigroups, that is, of all regular semigroups whose idempotents form a normal band. Let also $\mathbf{LGI}$ [$\mathbf{RGI}$] be the class of all left [right] generalized inverse semigroups, that is, of all regular semigroups whose idempotents form a left [right] normal band. The four classes of locally inverse semigroups introduced above are examples of existence varieties of regular semigroups [5].

We know from [22] that $\mathbf{I}$ satisfies (4.1). From both Lemma 3.6 and Theorem 3.9 of [15] we conclude that if S is a locally inverse semigroup, then the structure of the pseudosemilattice of idempotents of $T(S)$ depends only on the structure of the pseudosemilattice $E(S)$, and not on the entire structure of S. Thus, the existence varieties $\mathbf{ESLI}$, $\mathbf{RGI}$ and $\mathbf{LGI}$ also satisfy (4.1) due to results stated at the end of [2]. If S is a generalized inverse semigroup and $E, F \in T(S)$ are idempotents, then $EF = E \wedge F$ since $E(S)$ is a normal band for the semigroup operation, and thus $E(T(S))$ is a subband of $T(S)$, whence $E(T(S))$ is a normal subband of $T(S)$ and $T(S)$ is a generalized inverse semigroup. Therefore, the existence variety $\mathbf{GI}$ of all generalized inverse semigroups is another existence variety with property (4.1). In conclusion

Proposition 4.1. Any existence variety

$$\mathbf{V} \in \{\mathbf{I}, \mathbf{RGI}, \mathbf{LGI}, \mathbf{GI}, \mathbf{ESLI}, \mathbf{LI}\} \cup [\mathbf{T}, \mathbf{NBG}]$$

satisfies condition (4.1), where **T** denotes the existence variety of all trivial semigroups. Any other existence variety of locally inverse semigroups satisfying condition (4.1) must be contained in [**GI**, **ESLI**].

The question of which existence varieties from [**GI**, **ESLI**] have the property (4.1) is still open. Let **V** be an existence variety of completely simple semigroups containing the rectangular bands. Then $\mathbf{V} \vee \mathbf{I} \in$ [**GI**, **ESLI**]. Note also that **ESLI** is the join of **I** with the existence variety of all completely simple semigroups [3]. Maybe a first approach to answer this open question is to see under which conditions $\mathbf{V} \vee \mathbf{I}$ satisfies (4.1). It is not clear that the join of existence varieties satisfying (4.1) is another existence variety satisfying (4.1), and, in fact, this may well be not true. Even if we are able to characterize under which conditions $\mathbf{V} \vee \mathbf{I}$ has property (4.1), most certainly this will not solve this open question completely. In [3] an example is given of an existence variety of E-solid locally inverse semigroups which is not the join of an existence variety of completely simple semigroups with an existence variety of inverse semigroups. Although this example does not belong to [**GI**, **ESLI**], it is a strong indication that there exist existence varieties in [**GI**, **ESLI**] which cannot be written as $\mathbf{V} \vee \mathbf{I}$ with **V** an existence variety of completely simple semigroups.

References

[1] Auinger, K., The bifree locally inverse semigroup on a set, *J. Algebra* **166** (1994), 630–650.

[2] Auinger, K., On the lattice of existence varieties of locally inverse semigroups, *Canad. Math. Bull.* **37** (1994), 13–20.

[3] Auinger, K., J. Doyle, and P. R. Jones, On existence varieties of locally inverse semigroups, *Math. Proc. Camb. Philos. Soc.* **115** (1994), 197–217.

[4] Clifford, A. H., and G. B. Preston, *The Algebraic Theory of Semigroups, I*, American Mathematical Society, Providence, 1961; *II*, American Mathematical Society, Providence, 1967.

[5] Hall, T. E., Identities for existence varieties of regular semigroups, *Bull. Austral. Math. Soc.* **40** (1989), 59–77.

[6] Hall, T. E., Regular semigroups: amalgamation and the lattice of existence varieties, *Algebra Universalis* **29** (1991), 79–108.

[7] Hall, T. E., The concept of variety for regular semigroups, in "*Proceedings of the Monash Conference on Semigroups Theory*", T. E. Hall, P. R. Jones, J. Meakin eds., World Scientific, Singapore, 1991; 101–116.

[8] Higgins, P. M., *Techniques of Semigroup Theory*, Oxford Science Publications, Oxford University Press, Oxford, 1992.

[9] Howie, J. M., *Fundamentals of Semigroup Theory*, Clarendon Press, London, 1995.

[10] Kaďourek, J., and M. Szendrei, A new approach in the theory of orthodox semigroups, *Semigroup Forum* **40** (1990), 257–296.

[11] Lallement, G., Demi-groupes réguliers, *Ann. Mat. Pura Appl.* **77** (1967), 47–130.

[12] Munn, W. D., Semigroups satisfying minimal conditions, *Proc. Glasgow Math. Assoc.* (3) (1957), 145–152.

[13] Nambooripad, K. S. S., Pseudo-semilattices and biordered sets, I, *Simon Stevin* **55** (1981), 103–110; II, *Simon Stevin* **56** (1982), 143–159; III, *Simon Stevin* **56** (1982), 239–256.

[14] Pastijn, F. J., Regular locally testable semigroups as semigroups of quasi-ideals, *Acta Mathematica Acad. Sci. Hung.* **36** (1980), 161–166.

[15] Pastijn, F. J., and L. Oliveira, Maximal dense ideal extensions of locally inverse semigroups, *Semigroup Forum* **72** (2006), 441–458.

[16] Pastijn, F. J., and L. Oliveira, Ideal extensions of locally inverse semigroups, accepted in *Studia Sci. Math. Hungarica*.

[17] Pastijn, F. J., and L. Oliveira, Rees matrix covers and the translational hull of a locally inverse semigroup, preprint.

[18] Petrich, M., *Introduction to Semigroups*, Merrill, Columbus, 1973.

[19] Petrich, M., *Inverse Semigroups*, Wiley, New York, 1984.

[20] Petrich, M., The translational hull of a normal cryptogroup, *Math. Slovaca* **44** (1994), 245–262.

[21] Petrich, M., Bases for existence varieties of strict regular semigroups. *Bull. Belg. Math. Soc.* **8** (2001), 411–450.

[22] Ponizovskiĭ, I. S., *Zamechanie ob inversnykh polugruppakh*, Uspekhi Mat. Nauk **20** (1965), 147–148.

[23] Schein, B. M., Completions, translational hulls and ideal extensions of inverse semigroups, *Czechoslovak Math. J.* **23** (98) (1973), 575–610.

[24] Yeh, Y. T., The existence of e-free objects in e-varieties of regular semigroups, *Int. J. Algebra Comput.* **2** (1992), 471–484.

Contact information

F. J. Pastijn
Department of Mathematics, Statistics and Computer Science, Marquette University, Milwaukee, WI 53201-1881, USA
E-Mail: francisp@mscs.mu.edu

L. Oliveira
Departamento de Matemática Pura, Faculdade de Ciências da Universidade do Porto, R. Campo Alegre 687, 4169-007 Porto, Portugal
E-Mail: loliveir@fc.up.pt

Received by the editors: 11.05.2003
and in final form ??.??.????.