Algebra and Discrete Mathematics
Number 4. **(2006).** pp. 57 – 66 RESEARCH ARTICLE


# On groups with the minimal condition for non-invariant decomposable abelian subgroups

**F. N. Lyman and M. G. Drushlyak**

Communicated by L. A. Shemetkov

Abstract. The infinite groups, in which there is no any infinite descending chain of non-invariant decomposable abelian subgroups (*md*-groups) are studied. Infinite groups with the minimal condition for non-invariant abelian subgroups, infinite groups with the condition of normality for all decomposable abelian subgroups and others belong to the class of *md*-groups. The complete description of infinite locally finite and locally soluble non-periodic *md*-groups is given, the connection of the class of *md*-groups with other classes of groups are investigated.

The subgroup $H$ of the group $G$ will be called decomposable if it decomposes in the direct product of two non-trivial factors.

Infinite groups, in which there is no any infinite descending chain of non-invariant decomposable abelian subgroups, are studied in this article. Such groups will be called *md*-groups. Thus, if the *md*-group $G$ contains infinite descending chain of decomposable abelian subgroups $A_1 \supset A_2 \supset \ldots \supset A_n \supset \ldots$, then the subgroup $A_k$ is certainly invariant in the group $G$ for some natural number $k$.

The class of *md*-groups is wide enough. For example, infinite groups with such restrictions as the minimal condition for non-invariant abelian subgroups (Chernikov's $I$-groups [1]); the condition of normality for all infinite abelian subgroups (Chernikov's $IH$-groups [1]); the condition of normality for all decomposable abelian subgroups (in the case of non-abelian group such groups are called *di*-groups).

**Key words and phrases:** *group, subgroup, order of the group, involution, locally finite group, non-periodic group, decomposable abelian subgroup, minimal condition, condition of normality.*

All infinite groups, in which every subgroup does not decompose in the direct product of two non-trivial factors, concern to the *md*-groups. Such *md*-groups are described in [2] with condition of locally fineness in the periodic case and with the condition of locally solubility in the non-periodic case. From the theorems 1.1 and 2.1 of the article [2] will have following propositions, which concern *md*-groups without decomposable subgroups.

**Theorem 1.** *In an infinite locally finite group $G$ all abelian subgroups are indecomposable if and only if it is the group of the one of the following types:*

1) *$G$ is quasicyclic $p$-group for some prime $p$;*

2) *$G = A\langle b\rangle$, where $A$ is quasicyclic 2-group, $|b| = 4$, $b^2 \in A$ and $b^{-1}ab = a^{-1}$ for every element $a \in A$;*

3) *$G = A\lambda\langle b\rangle$ is the Frobenius' group, where $A$ is quasicyclic $p$-group, $B$ is cyclic $q$-group, $p$ and $q$ are prime, $(p-1, q) = q$.*

**Theorem 2.** *In a locally soluble non-periodic group $G$ all abelian subgroups are indecomposable if and only if it is the group of one of the following types:*

1) *$G$ is abelian torsion free group of rang 1;*

2) *$G = A\lambda B$ is the Frobenius' group, where $A$ is abelian torsion free group rang 1, $|b| = 2$ or $|b| = \infty$.*

Since the periodic indecomposable abelian group is cyclic or quasicyclic $p$-groups for some prime $p$, then the minimal condition for non-invariant decomposable abelian subgroups is equal to the minimal condition for non-invariant abelian subgroups in the periodic case. Groups with such restrictions were studied by Chernikov (see [1], theorem 4.11). That is why only non-periodic *md*-groups are studied further.

**Theorem 3.** *A non-periodic group $G$, which does not satisfy the minimal condition for decomposable abelian subgroups, satisfies the minimal condition for non-invariant decomposable abelian subgroups if and only if it is the group of one of the following types:*

1) *$G$ is non-periodic abelian groups with decomposable subgroups;*

2) *$G = Q \times B$, where $Q$ is the quaternion group of order 8, $B$ is abelian torsion free group of rank 1;*

3) *$G = \langle x \rangle \lambda A$, where $|x| = p^n$, $p$ is prime ($p = 2$, $n > 1$), $A$ is non-divisible abelian torsion free group of rank 1 and commutant $G'$ is of prime order;*

4) *$G = \left(\left\langle x^2 \right\rangle \times A\right) \langle x \rangle$, where $|x| = 8$, $A$ is abelian torsion free group of rank 1 and quotient group $G/\left\langle x^4 \right\rangle$ is $IH$-group;*

5) *$G = (\langle z \rangle \times A) \lambda \langle x \rangle$, where $|z| = 4$, $|x| = 2$, $A$ is abelian torsion free group of rank 1 and quotient group $G/\left\langle z^2 \right\rangle$ is $IH$-group;*

6) *$G = A \langle b \rangle$, where $A$ is non-periodic abelian group, $b^4 = 1$ and $b^{-1}ab = a^{-1}$ for an arbitrary element $a \in A$, the centre $Z(G)$ is of order $2^n$, $n \geq 0$ and when $|b| = 2$ the group $A$ contains decomposable subgroups;*

7) *$G = A\lambda \langle b \rangle$, where $A$ is non-periodic abelian group, which does not contain free abelian subgroups of rank 2, or non-abelian di-group with infinite centre, $|b| = 2$, the torsion part $T(A)$ is 2-group with one involution $\langle a \rangle$ and $G/\langle a \rangle$ is $IH$-group;*

8) *$G = C\lambda \langle x \rangle$, where $C = C_G(a)$, $|a| = \infty$, $\langle a \rangle \lhd G$, $|Z(G)| = p$ and $G/Z(G)$ - $IH$-group;*

9) *$G = (\langle b \rangle \lambda A) \lambda \langle x \rangle$, where $|b| = p \neq 2$, $A = \langle a \rangle A_1$ is abelian torsion free group of rank 1, $a^2 \in A_1$, $[A_1, b] = [x, b] = 1$, $a^{-1}ba = b^{-1}$, $x^{-1}yx = y^{-1}$ for an arbitrary element $y \in A$.*

*Proof.* It is evident that the groups of every type which is enumerated in the condition of the theorem satisfy the minimal condition for non-invariant decomposable abelian subgroups. That is why we will prove only the necessity of the conditions of the theorem. The proof of necessity is completed by lemmas, which are proved further.

**Lemma 1.** *If a non-periodical abelian subgroup $A$ of the $md$-group $G$ contains free abelian subgroups of rank 2 or decomposable periodic subgroup, then all subgroup of $A$ are invariant in the group $G$. If $A = \langle a \rangle \times \langle b \rangle$, where $|a| = \infty$, $1 < |b| < \infty$, then $\langle a, b \rangle \lhd G$, all periodic subgroups of $A$ are invariant in the group $G$.*

*Proof.* Suppose that $A \supset \langle a \rangle \times \langle b \rangle$, where $|a| = \infty$. Hence according to the definition of the $md$-group the invariant subgroup $\left\langle a^{p^m}, b \right\rangle$ exists in the group $G$ among the subgroup of the chain $\langle a, b \rangle \supset \langle a^p, b \rangle \supset \left\langle a^{p^2}, b \right\rangle \supset \ldots \supset \left\langle a^{p^n}, b \right\rangle \supset \ldots$, where $p$ is prime. Also the invariant subgroup $\left\langle a^{q^k}, b \right\rangle$ ($q \neq p$) exists in the group $G$. Since $\left(p^m, q^k\right) = 1$, then there are two integer numbers $u$ and $v$, such as $p^m u + q^k v = 1$. Then $\left\langle a^{p^m}, a^{q^k} \right\rangle = \langle a \rangle$, $\left\langle a^{p^m}, b \right\rangle \cdot \left\langle a^{q^k}, b \right\rangle = \langle a, b \rangle \lhd G$.

If $1 < |b| < \infty$, $\langle b \rangle \lhd G$ comes from the condition $\langle a, b \rangle \lhd G$. Thus, all periodic subgroups of the mixed abelian subgroup $A$ of the *md*-group $G$ and it itself are invariant in $G$. Let $A \supset \langle c \rangle \times \langle d \rangle$, where $\langle c \rangle$ and $\langle d \rangle$ are finite non-trivial subgroups. Then $\langle y, c \rangle \lhd G$, $\langle y, d \rangle \lhd G$ for an arbitrary element $y \in A$ and that is why $\langle y, c \rangle \cap \langle y, d \rangle = \langle y \rangle \lhd G$.

Let $A \supset \langle a, b \rangle$, where $|a| = |b| = \infty$. Then there is infinite cyclic subgroup $\langle a_1 \rangle \subset \langle a, b \rangle$ for an arbitrary element $y \in A$ such as $\langle a_1 \rangle \cap \langle y \rangle = 1$. Then according to proved earlier we have $\langle a_1^n, y \rangle \lhd G$ for every natural number $n$. Hence $\cap_{n=1}^{\infty} \langle a_1^n, y \rangle = \langle y \rangle \lhd G$. Thus, all subgroups of $A$ are invariant in the group $G$, if the subgroup $A$ contains the direct product of two finite or two infinite cyclic subgroups. This completes the proof of the lemma.

**Lemma 2.** *All infinite cyclic subgroups are invariant in the non-periodic non-abelian group $G$ if and only if $G = C \langle b \rangle$, where $C$ is non-periodic abelian subgroup, $b^4 = 1$ and $b^{-1}ab = a^{-1}$ for an arbitrary element $a \in C$.*

*Proof.* This lemma was given without proof in the article [3]. Let us pay attention at the necessary conditions of the lemma, because theirs sufficiency is evident. Let $G$ be non-periodic non-abelian group, in which all infinite cyclic subgroups are invariant. Let us show, that the centralizer $C$ of an arbitrary elemen $x$ of infinite order of the group $G$ is abelian subgroup. If $y \in C$, $z \in C$, $|y| = |z| = \infty$, then $\langle y \rangle \lhd G$, $\langle z \rangle \lhd G$ and that is why $[y, z] = 1$. If $c \in C$ and $|c| < \infty$, then $|xc| = \infty$ and $[y, xc] = [y, c] = 1$. If $c_1 \in C$, $c_2 \in C$ and $|c_1| < \infty$, $|c_2| < \infty$, then $|c_1 x| = |c_2 x| = \infty$ and $[c_1 x, c_2 x] = [c_1, c_2] = 1$. Consequently, the subgroup $C$ is abelian.

It is evident, that the subgroup $C$ contains all elements of infinite order and all elements of finite odd order of the group $G$. That is why $G = C \langle b \rangle$, $b^{2^n} = 1$, $n \geq 1$ and the element $b$ is not commutative with every element of infinite order of the group $G$. Thus, if $y \in C$, $|y| = \infty$, then $b^{-1}yb = y^{-1}$. If $c \in C$ and $|c| < \infty$, then $|cy| = \infty$ and $b^{-1}(cy)b = (cy)^{-1} = c^{-1}y^{-1}$, $b^{-1}cb = c^{-1}$. At last, from the equalities $b^{-1}(yb^2)b = (yb^2)^{-1} = y^{-1}b^2$ we get $b^4 = 1$. This completes the proof of the lemma.

The corollary of this lemma is the Chernikov's theorem (see [1], theorem 4.6) on the structure of non-periodic $IH$-groups - non-abelian non-periodic groups, in which all infinite abelian subgroups are invariant.

**Corollary.** *Non-periodic non-abelian group $G$ is $IH$-group if and only if it has the finite centre of the expornent, which is not more than 2, and all its infinite cyclic subgroups are invariant.*

**Lemma 3.** *A non-periodic md-group $G$ is abelian if its centre contains the free abelian subgroup of rank 2 or the decomposable abelian subgroup of order $pq \neq 4$, where $p$ and $q$ are prime.*

*Proof.* Let the centre $Z(G)$ of *md*-group $G$ contains the free abelian subgroup $\langle a\rangle \times \langle b\rangle$. Then all subgroups from $\langle a, b, g\rangle$ are invariant in $G$ for an arbitrary element $g \in G$ by lemma 1. That is why $\langle g\rangle \lhd G$ and the group $G$ is abelian.

Let $Z(G) \supset \langle c\rangle \times \langle d\rangle$, where the elements $c$ and $d$ are of finite order and if only one of them is of odd order. Then by lemma 1 we have $\langle y, c\rangle \lhd G$, $\langle y, d\rangle \lhd G$ for an arbitrary element $y \in G$ of infinite order and hence $\langle y\rangle \lhd G$. Thus, all infinite cyclic subgroups are invariant in the group $G$. Since $expZ(G) > 2$, then the group $G$ is abelian by the lemma 2. This completes the proof of the lemma.

**Lemma 4.** *If a md-group $G$ contains the periodic abelian subgroup $A$, which does not satisfy the minimal condition, then all subgroups of $A$ are invariant in the group $G$.*

*Proof.* According to the condition of the lemma, the subgroup $A$ contains the direct product $A_1 = \langle a_1\rangle \times \langle a_2\rangle \times ... \times \langle a_n\rangle \times ...$ of the infinite quantity of subgroups $\langle a_i\rangle$, $i = 1, 2, ..., n, ...$ of finite order. Let $\langle x\rangle$ be an arbitrary cyclic subgroup of $A$. Then there are such infinite subgroups $B \subset A_1$, $C \subset A_1$, that $\langle x\rangle \cap B = \langle x\rangle \cap C = B \cap C = 1$. Let $B = \langle b_1\rangle \times \langle b_2\rangle \times ... \times \langle b_n\rangle \times ...$, $C = \langle c_1\rangle \times \langle c_2\rangle \times ... \times \langle c_n\rangle \times ...$. Then according to the definition of the *md*-groups every chain $\langle x, b_1, b_2, ..., b_n, ...\rangle \supset \langle x, b_2, ..., b_n, ...\rangle \supset ...$, $\langle x, c_1, c_2, ..., c_n, ...\rangle \supset \langle x, c_2, ..., c_n, ...\rangle \supset ...$ contains the invariant subgroup. Let it be respectively subgroups $\langle x, b_k, ..., b_n, ...\rangle$ and $\langle x, c_m, ..., c_n, ...\rangle$. Their intersection is equal to $\langle x\rangle$ and that is why $\langle x\rangle \lhd G$. Hence all subgroups of $A$ are invariant in $G$. This completes the proof of the lemma.

**Lemma 5.** *A non-abelian md-group $G$ is the group of one of the types 2)-3) of the theorem 3, if it contains decomposable subgroups and has non-periodic centre.*

*Proof.* Let $z \in Z(G)$, $|z| = \infty$ and $B$ be an arbitrary decomposable abelian subgroup of the group $G$. If the subgroup $B$ is non-periodic, then $B \lhd G$ by the lemma 1. If the subgroup $B$ is periodic, then $(\langle z\rangle \times B) \lhd G$ by the lemma 1 and hence $B \lhd G$. Thus, all decomposable abelian subgroups are invariant in the group $G$. From the description of such groups in [2] we have that the group $G$ is the group of the one of the following types 2)-3) of the theorem 3. This completes the proof of the lemma.

**Lemma 6.** *A non-periodic md-group $G$, which does not satisfy the minimal condition for decomposable abelian subgroups, contains the invariant infinite cyclic subgroup. The centralizer of every infinite cyclic invariant subgroup contains all elements of infinite order.*

*Proof.* Let a non-periodic *md*-group $G$ does not satisfy the minimal condition for decomposable abelian subgroups. Then $G$ has finite chain $A_1 \supset A_2 \supset ... \supset A_n \supset ...$ of abelian decomposable subgroups. There is the subgroup $A_i \lhd G$ according to the definition of the *md*-group. Since $A_i$ does not satisfy the minimal condition, then the group $G$ contains the invariant infinite cyclic subgroup by lemmas 1 and 4.

Let $\langle a\rangle \lhd G$, $|a| = \infty$, $x \in G$, $|x| = \infty$. Let us show that $x \in C_G(a)$. Suppose, that $[a, x] \neq 1$. Then $x^{-1}ax = a^{-1}$, $\langle x\rangle \cap \langle a\rangle = 1$, $x^2 \in C_G(a)$. By the lemma 1 all subgroups from $\left\langle a, x^2\right\rangle$ are invariant in the group $G$. Then $\left[x, ax^2\right] = [x, a] \neq 1$ and that is why $x^{-1}(ax^2)x = (ax^2)^{-1} = a^{-1}x^2$. Hence $a^{-1}x^{-2} = a^{-1}x^2$, $x^4 = 1$. We obtain a contradiction. Thus $[a, x] = 1$. This completes the proof of the lemma.

Further let us investigate a non-periodic *md*-groups with the periodic centre.

**Lemma 7.** *If a non-periodic md-group $G$ has the periodic centre $Z(G)$ of even order, then $G = C\langle x\rangle$, where $C$ is the centralizer of every invariant infinite cyclic subgroup of $G$, $x^8 = 1$. $expZ(G) = 2^m \leq 4$ and $|Z(G)| < \infty$. If $expZ(G) = 4$, then $Z(G)$ is the cyclic group.*

*Proof.* By the lemma 6 the group $G$ has invariant infinite cyclic subgroup $\langle a\rangle$. Then $G = C\langle x\rangle$, where $C = C_G(a)$, $x \notin C$, $x^2 \in C$, $|x| < \infty$. $|x| = 2^n$, $n \geq 1$ and $x^{-1}ax = a^{-1}$. Let $|x| > 2$. By the lemma 1 the subgroup $\left\langle ax^2\right\rangle \times \left\langle x^{2^{n-1}}\right\rangle$ is invariant in $G$. Then $\left\langle ax^2, x^{2^{n-1}}\right\rangle^2 = \left\langle a^2x^4\right\rangle \lhd G$. Hence $x^{-1}(a^2x^4)x = (a^2x^4)^{-1} = a^{-2}x^{-4} = a^{-2}x^4$, $x^{-4} = x^4$, $x^8 = 1$.

Let us find out, elements of what order can be in the centre $Z(G)$. Let $z \in Z(G)$, $|z| < \infty$. According to the condition of the lemma $Z(G)$ contains the involution $i$. Then $(\langle az\rangle \times \langle i\rangle) \lhd G$, $\left\langle a^2z^2\right\rangle \lhd G$, $x^{-1}a^2z^2x = (a^2z^2)^{-1} = a^{-2}z^2$, $z^4 = 1$.

We have, that $|Z(G)| < \infty$, because else $Z(G)\langle x\rangle$ is infinite abelian 2-group, which does not satisfy the minimal condition. Then we have $\langle x\rangle \lhd G$ by the lemma 4. Hence we obtain the contradiction according to the condition of the lemma.

Let $expZ(G) = 4$. Let us show, that in this case $Z(G) = \langle z\rangle$. Suppose, that $Z(G) \supset \langle z\rangle \times \langle c\rangle$, where $|z| = 4$, $|c| = 2$. Then $(\langle az\rangle \times \left\langle z^2\right\rangle) \lhd G$, $(\langle az\rangle \times \langle c\rangle) \lhd G$. Hence $\left\langle az, z^2\right\rangle \cap \langle az, c\rangle = \langle az\rangle \lhd G$. Then $x^{-1}(az)x = (az)^{-1} = a^{-1}z^{-1} = a^{-1}z$. And $z^2 = 1$, we obtain the contradiction according to the condition of the lemma. This completes the proof of the lemma.

**Lemma 8.** *If a non-periodic md-group $G$ has the cyclic centre of order 4, then $G$ is the group of the one of the types 4)-5)of the theorem 3.*

*Proof.* By the lemma 7 we have $G = C\langle x\rangle$, where $C = C_G(a)$,

$|a| = \infty$, $\langle a \rangle \lhd G$, $x^8 = 1$. Let $|x| = 8$. Then $x^2 \in C$. Let us show, that $Z(G) = \langle x^2 \rangle$. Since $expZ(G) = 4$, then the group $G$ contains the non-trivial infinite cyclic subgroup by the lemma 2. The fact, that the subgroup $C$ does not contains the free abelian subgroup of rank 2 and the decomposable periodic subgroup, follows from the lemma1. That is why the torsion part $T(C)$ is locally cyclic 2-group or the quaternion group and all subgroups of $T(C)$ are invariant in $G$. Since $\langle y, x^4 \rangle \lhd G$ for an arbitrary element $y \in G$ of infinite order, then all infinite cyclic subgroups are invariant in quotient group $G/\langle x^4 \rangle$ and its centre is finite. Hence $G/\langle x^4 \rangle$ is $IH$-group.

Let us specify the structure of the subgroup $C$. Let $y_1, y_2$ are arbitrary elements of infinite order of the group $G$. Then $\langle y_1, y_2, x^4 \rangle$ is abelian group, because $\langle y_1, y_2, x^4 \rangle / \langle x^4 \rangle$ is cyclic group. Hence $C$ is abelian group and $C = \langle x^2 \rangle \times A$, where $A$ is abelian torsion free group of rank 1. The group $G$ is the group of the type 4) of the theorem 3.

Let $Z(G) = \langle z \rangle$, $|z| = 4$ and $|x| \leq 4$. If $|x| = 4$, then $\langle x \rangle \cap \langle z \rangle \neq 1$, else all infinite cyclic subgroups are invariant in $G$. We obtain the contradiction according to the condition of the lemma. Then $(xz)^2 = 1$ and let us take the element $xz$ instead the element $x$. Hence, $|x| = 2$. Also $C = \langle z \rangle \times A$, where $A$ is abelian torsion free group of rank 1. The group $G$ is the group of the type 5) of the theorem 3. This completes the proof of the lemma.

Further let us study non-periodic $md$-groups, which have $expZ(G) = 2$ and $4 \leq |Z(G)| < \infty$.

**Lemma 9.** *If the centre of non-periodic md-group $G$ is finite elementary 2-subgroup of the order, which is not less then 4, then $G$ is $IH$-group and belongs to the groups of the type 6) of the theorem 3.*

*Proof.* Let $Z(G) \supset \langle z_1 \rangle \times \langle z_2 \rangle$, $|z_1| = |z_2| = 2$, $y \in G$ and $|y| = \infty$. Then $\langle y, z_1 \rangle \lhd G$, $\langle y, z_2 \rangle \lhd G$ by lemma 1. Hence $\langle y \rangle \lhd G$. Since $|Z(G)| < \infty$, then $G$ is $IH$-group and according to the corollary of the lemma 2 $G$ is the group of the type 6) of the theorem 3. This completes the proof of the lemma.

**Lemma 10.** *If a non-periodic md-group $G = C\langle x \rangle$, where $C = C_G(a)$, $|a| = \infty$ and $\langle a \rangle \lhd G$, $|x| \leq 4$, $|Z(G)| = 2$, then either $G$ is $IH$-group or $G/Z(G)$ is $IH$-group. $G$ is the group of either the type 6) or the type 7) of the theorem 3.*

*Proof.* If the subgroup $C$ contains the free abelian subgroup of rank 2 or the decomposable periodic subgroup, then all infinite cyclic subgroups of the group $G$ are invariant. Since $|Z(G)| = 2$, then $G$ is $IH$-group according to the corollary of the lemma 2 and $G$ belongs to the groups of the type 6) of the theorem 3.

Let the torsion part $T(C)$ does not contain any decomposable subgroups and $C$ does not contains free abelian subgroups of rank 2. Then $T(C)$ is 2-group with one involution and that is why it is either the locally cyclic 2-group or the quaternion group. It is either $\langle y\rangle \lhd G$ for an arbitrary element $y \in G$ of infinite order or there is such element $y \in G$, that $|y| = \infty$, $\langle y\rangle$ is not invariant in $G$, but $\langle y, i\rangle \lhd G$, where $\langle i\rangle = Z(G)$.

In the first case we also have, that $G$ is the group of the type 6) of the theorem 3.

In the other case the quotient group $\bar{G} = G/\langle i\rangle$ is $IH$-group. All decomposable subgroups of the group $G$ are non-periodic, are contained in $C$, are invariant in $G$, if $T(C)\langle x\rangle$ has one involution.

Suppose, that $T(C)\langle x\rangle$ contains the elementary abelian subgroup $A$ of order 4. Then, when $|x| = 4$ $G \supset \langle x\rangle \times \langle i\rangle$, where $i^2 = 1$, we have $\langle i\rangle = Z(G)$. Since $\left[x^2, a\right] = 1$, then $\left\langle x^2\right\rangle \lhd G$, $x^2 \in Z(G)$, and we obtain a contradiction. Hence $|x| = 2$ and $G$ is the group of the type 7) of the theorem 3. This completes the proof of the lemma.

**Lemma 11.** *If a non-trivial centre $Z(G)$ of the non-periodic md-group $G$ is periodic subgroup with the elements of odd order, then $|Z(G)| = p$, $p$ is prime odd number.*

*Proof.* The centre $Z(G)$ is the locally cyclic $p$-group for some odd prime $p$ according to the lemma 3. Suppose, that $Z(G) \supset \langle b\rangle$, $|b| = p^2$. Then we have $\langle ab, b^p\rangle \lhd G$, $\langle a^p, b^p\rangle \lhd G$, $x^{-1}a^pb^px = (a^pb^p)^{-1} = a^{-p}b^p$, $b^{-p} = b^p$, $b^{2p} = 1$ for subgroup $\langle a\rangle$, where $|a| = \infty$ and $\langle a\rangle \lhd G$, $x \notin C_G(a)$. We obtain a contradiction. Thus, $|Z(G)| = p$. This completes the proof of the lemma.

**Lemma 12.** *If the centre $Z(G)$ of the non-periodic md-group $G$ is of the order $p \neq 2$, then $G = C\lambda\langle x\rangle$, $|x| = 2$, $C = C_G(a)$, $|a| = \infty$, $\langle a\rangle \lhd G$, $G/Z(G)$ is $IH$-group and the group $G$ belongs to the groups of the type 8) of the theorem 3.*

*Proof.* $\langle y, Z(G)\rangle \lhd G$ for an arbitrary element $y \in G$ of infinite order. Then all infinite cyclic subgroups are invariant in $\bar{G} = G/Z(G)$. Since $\bar{G} = \bar{C}\lambda\langle\bar{x}\rangle$ and $\bar{C}$ has not involution, then $Z(\bar{G}) = 1$, $\bar{G}$ is $IH$-group. The group $G$ is the group of the type 8) of the theorem 3. This completes the proof of the lemma.

Let us investigate the structure of the non-periodic $md$-groups, which does not satisfy the minimal condition for decomposable abelian subgroups and has trivial centre. In this case we have $G = C\lambda\langle x\rangle$, where $C = C_G(a)$, $|a| = \infty$, $\langle a\rangle \lhd G$, $|x| = 2$ and the subgroup $C$ is either abelian or $di$-group.

**Lemma 13.** *If a non-periodic md-group $G$ with the trivial centre does not satisfy the minimal condition for the decomposable abelian subgroups and $G = C\lambda\langle x\rangle$, where $C = C_G(a)$ is abelian group, $|a| = \infty$, $\langle a\rangle \lhd G$, $|x| = 2$, then $G$ is $IH$-group and belongs to the groups of the type 6) of* *the theorem 3.*

*Proof.* If the subgroup $C$ contains the decomposable abelian either torsion free subgroup or periodic subgroup, then all subgroups of $C$ are invariant in $G$ by the lemma 1. Since $Z(G)=1$, then $G$ is $IH$-group and belongs to the groups of the type 6) of the theorem 3.

Suppose, that $C$ does not contain decomposable torsion free subgroups and periodic decomposable subgroups. Then $C=A\times B$, where $A$ is the abelian torsion free group of rank 1, $B$ is the locally cyclic $p$-group, $p\neq 2$. Let $b\in B$, $|b|=p$. According to the proved in the lemma 1 we have $B\lhd G$ and $\langle b,y\rangle\lhd G$ for an arbitrary element $y\in G$ of infinite order. Let $\langle y\rangle$ is not invariant in $G$. Then $x^{-1}yx=y^kb^m$, $x^{-1}y^px=y^{pk}=y^{-p}$. Hence $k=-1$. Since $[x,b]\neq 1$, then $x^{-1}bx=b^n$, $n\neq 1$. $b=x^{-2}bx^2=(x^{-1}bx)^n=b^{n^2}$. That is why $n^2\equiv 1(modp)$ and $n=-1$, $x^{-1}bx=b^{-1}$. Then $y=x^{-2}yx^2=x^{-1}(y^{-1}b^m)x=(x^{-1}yx)^{-1}(x^{-1}bx)^m=y^{-k}b^{-m}$, $b^{-m}=yb^{-2m}$.

Hence $b^{-2m}=1$, $b^m=1$ and $x^{-1}yx=y^{-1}$. Thus, $\langle y\rangle\lhd G$. Hence all infinite cyclic subgroups are invariant in $G$ and $Z(G)=1$. Then $G$ is $IH$-group and belongs to the groups of the type 6) of the theorem 3 according to the corollary of the lemma 2. This completes the proof of the lemma.

**Lemma 14.** *If a non-periodic md-group $G$ with the trivial centre does not satisfy the minimal condition for decomposable abelian subgroups and $G=C\lambda\langle x\rangle$, where $C=C_G(a)$ is abelian di-group, $|a|=\infty$, $\langle a\rangle\lhd G$, $|x|=2$, then $G$ is the group of the type 9) of the theorem 3.*

*Proof.* Since the centre $Z(G)$ is non-periodic and all finite subgroups of $C$ are invariant in $G$ and $Z(G)=1$, then $C=\langle b\rangle\lambda A$ by the theorem 2.2. from [1], where $|b|=p^n$, $p\neq 2$, $n\geq 1$, $A$ is non-divisible abelian torsion free subgroup of rank 1 and the commutant $C'$ is of prime order $p$. If the subgroup $C$ contains all elements of infinite order of the group $G$, all $p$-elements $(p\neq 2)$ and the group $G$ does not contain elements of order 4, then $(xb)^2=1$. Hence $x^{-1}bx=b^{-1}$.

Let us show, that $|b|=p$. Let $|b|=p^2>p$. Then $b^{p^{n-1}}=b_1\in Z(G)$. Then $\langle b_1,y\rangle\lhd G$ for an arbitrary element $y\in G$ of infinite order by the lemma 1. Hence $\langle y^p\rangle\lhd G$ and that is why $x^{-1}yx=y^kb_1^m$, $x^{-1}y^px=y^{kp}=y^{-p}$. Consequensly, $k=-1$. Further we have $y=x^{-2}yx^2=x^{-1}(y^{-1}b_1^m)x=(x^{-1}yx)^{-1}\cdot(x^{-1}b_1x)^m=yb_1^{-m}\cdot b_1^{-m}=yb_1^{-2m}$.

Since $p\neq 2$, then $b_1^m=1$, $x^{-1}yx=y^{-1}$. Let us take an arbitrary element $a\in G$. Then $|ba|=\infty$ and according to the proved $x^{-1}(ba)x=(ba)^{-1}=a^{-1}b^{-1}$ and $x^{-1}(ba)x=(x^{-1}bx)\cdot(x^{-1}ax)=b^{-1}a^{-1}$. Hence $[a,b]=1$. We obtain the contradiction according to the condition of the lemma. Thus, $|b|=p\neq 2$.

The group $G$ contains invariant decomposable abelian subgroups by the theorem 2.2. from [1]. Such group can be only the group of order $2p$ by the lemma 1. That is why $[x, b] = 1$.

Let $C_A(b) = A_1 \neq A$. Then $A/A_1$ is finite cyclic group and that is why $A = \langle a \rangle A_1$ for some element $a \in G$.

Let us investigate the possible order of the element $xa$. Since the subgroup $C$ contains all elements of infinite order and all $p$-elements, then either $|xa| = 2p$ or $|xa| = 2$. If $|xa| = 2p$, then $\left|(xa)^2\right| = p$ and $(xa)^2 = b^k$, $(k, p) = 1$. Hence $x^{-1}ax = b^k a^{-1}$. Then $a = x^{-2}ax^2 = x^{-1}(b^k a^{-1})x = b^k \cdot (x^{-1}ax)^{-1} = b^k ab^{-k}$ and that is why $a \in C_A(b)$. Hence we obtain the contradiction according to the condition of the lemma. Thus, $|xa| = 2$ and $x^{-1}ax = a^{-1}$. Hence $x^{-1}yx = y^{-1}$ for an arbitrary element $y \in A$.

Let $C_1 = C_G(b)$. It is evident, that $C_1 = \langle b, A_1, x \rangle \lhd G$. Then $[a, x] = a^{-2} \in C_1$. That is why $a^{-1}ba = b^{-1}$. At last we have $G = (\langle b \rangle \leftthreetimes \langle a \rangle A_1) \leftthreetimes \langle x \rangle$, $|b| = p \neq 2$, $|x| = 2$, $\langle a \rangle A_1 = A$ is non-divisible abelian torsion free group of rank 1, $a^2 \in A_1$, $[A_1, b] = [x, b] = 1$, $a^{-1}ba = b^{-1}$, $x^{-1}yx = y^{-1}$ for an arbitrary element $y \in A$. Thus, the group $G$ is the group of the type 9) of the theorem 3. This completes the proof of the lemma. This also completes the proof of the theorem.

**Corollary.** *A non-periodic md-group $G$, which contains decomposable abelian subgroups, contains non-invariant decomposable abelian subgroups if and only if, it is the group of the type 5),7), 8), 9) of the theorem 3 or the group of the type 6) of the theorem 3, if it contains the Klein's subgroup.*

## References

[1] Lyman F.M. Groups, in which all decomposable subgroups are invariant // Ukr. Math.J. - 1970. - V.22, N.6. - P.725-733.

[2] Chernikov S.N. Groups with defined properties of the system of subgroups. - Moscow: Nauka, 1980. - 384p.

[3] Lyman F.M. Non-periodic groups, in which all decomposable *pd*-subgroups are invariant // Ukr. Math.J. - 1988. - V.40, N.3. - P.330-335.

CONTACT INFORMATION

**F. N. Lyman** Full postal address of the first author
*E-Mail:* f.author@domain.com

**M. G. Drushlyak** Full postal address of the second author
*E-Mail:* s.author@domain.com
*URL:* www.sauthor.domain.com

Received by the editors: 11.08.2006
and in final form ??.??.????.