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RESEARCH ARTICLE

Value distribution of general Dirichlet series. VIII

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Abstract. A joint limit theorem on the complex plane for a new class of general Dirichlet series is proved.

1. Introduction

Let $s = \sigma + it$ be a complex variable, $\{a_m : m \in \mathbb{N}\}$ be a sequence of complex numbers, and let $\{\lambda_m : m \in \mathbb{N}\}$ be an increasing sequence of positive numbers, $\lim_{m\to\infty} \lambda_m = +\infty$. The series of the form

$$f(s) = \sum_{m=1}^{\infty} a_m \mathrm{e}^{-\lambda_m s} \tag{1}$$

is called a general Dirichlet series. If $\lambda_m = \log m$, we obtain the ordinary Dirichlet series

$$\sum_{m=1}^{\infty} \frac{a_m}{m^s}.$$

It is well known that the region of convergence as well as of absolute convergence of Dirichlet series is a half-plane.

The first probabilistic results for Dirichlet series were obtained by H.Bohr and B.Jessen. In [2] and [3] they proved theorems for the Riemann zeta-function which are similar to modern limit theorems in the sense of weak convergence of probability measures. The investigations

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of H.Bohr and B.Jessen were developed and generalized by A. Wintner, V.Borchsenius, A.Selberg, P.D.T.A. Elliott, A.Ghosh, K.Matsumoto, B.Bagchi, E.M.Nikishin, E.Stankus, J.Steuding, W.Schwarz, the author and others. The results of such a kind can be found in [7], [8], [14] and [20].

Limit theorems in the sense of weak convergence of probability measures in various spaces for general Dirichlet series were obtained [4]-[6], [10]-[14] and [18], [19]. Limit theorems on the complex plane for general Dirichlet series were proved in [12]-[14]. Denote by $meas\{A\}$ the Lebesgue measure of a measurable set $A \in \mathbb{R}$, and let, for $T > 0$,

$$\nu_T(...) = \frac{1}{T}\,\text{meas}\{t \in [0,T] : ...\},$$

where in place of dots a condition satisfied by t is to be written. Moreover, let $\mathcal{B}(S)$ be the class of Borel sets of the space S.

Denote by γ the unit circle $\{s \in \mathbb{C} : |s| = 1\}$ on the complex plane $\mathbb{C}$, and define

$$\Omega = \prod_{m=1}^{\infty} \gamma_m,$$

where $\gamma_m = \gamma$ for each $m \in \mathbb{N}$. Then the infinite-dimensional torus Ω in view of the Tikhonov theorem is a compact topological Abelian group, therefore the probability Haar measure m_H on $(\Omega, \mathcal{B}(\Omega))$ can be defined. This gives a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(m)$ the projection of $\omega \in \Omega$ to the coordinate space γ_m, $m \in \mathbb{N}$.

Suppose that the series (1) converges absolutely for $\sigma > \sigma_a$. Then the function $f(s)$ is analytic in the half-plane $\{s \in \mathbb{C} : \sigma > \sigma_a\}$. Moreover, we require that the function $f(s)$ should be meromorphically continuable to the half-plane $\{s \in \mathbb{C} : \sigma > \sigma_1\}$, $\sigma_1 < \sigma_a$, all poles being included in a compact set, and that, for $\sigma > \sigma_1$, the estimates

$$f(\sigma + it) \ll |t|^{\alpha}, \quad \alpha > 0, \quad |t| \geqslant t_0 > 0, \tag{2}$$

and

$$\int_{-T}^{T} |f(\sigma + it)|^2 \, \mathrm{d}\, t \ll T, \quad T \to \infty, \tag{3}$$

should be satisfied. Suppose that the exponents λ_m satisfy the inequality

$$\lambda_m \geqslant (\log m)^{\delta} \tag{4}$$

with some positive $\delta > 0$. Then in [12] it was proved that under the last conditions, for $\sigma > \sigma_1$,

$$f(\sigma, \omega) = \sum_{m=1}^{\infty} a_m \omega(m) \mathrm{e}^{-\lambda_m \sigma},$$

is a complex-valued random variable defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. If the system $\{\lambda_m\}$ is linearly independent over the field of rational numbers, then it was obtained in [12] that, for $\sigma > \sigma_1$, the probability measure

$$\nu_T\big(f(\sigma+it) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}), \tag{6}$$

converges weakly to the distribution of the random variable $f(\sigma, \omega)$ as $T \to \infty$.

Condition (4) is rather strong, it limits a class of Dirichlet series for which a limit theorem is true. Suppose that, for $\sigma > \sigma_1$,

$$\sum_{m=1}^{\infty} |a_m|^2 \mathrm{e}^{-2\lambda_m \sigma} \log^2 m < \infty. \tag{7}$$

Then in [14] the following statement has been obtained.
Theorem A. *Suppose that the system $\{\lambda_m\}$ is linearly independent over the field of rational numbers, and conditions (2), (3) and (7) are satisfied. Then the probability measure (6) converges weakly to the distribution of the random element $f(\sigma, \omega)$ as $T \to \infty$.*

In [13] a joint limit theorem on the complex plane for general Dirichlet series was proved. Let, for $\sigma > \sigma_{aj}$,

$$f_j(s) = \sum_{m=1}^{\infty} a_{mj} \mathrm{e}^{-\lambda_{mj} s},$$

where $\{a_{mj}\}$ and $\{\lambda_{mj}\}$ are a sequence of complex numbers and an increasing sequence of positive numbers, $\lim\limits_{m\to\infty} \lambda_{mj} = +\infty$, respectively, $j = 1, ..., r$, $r > 1$. Suppose that the function $f_j(s)$ is meromorphically continuable to the region $\{s \in \mathbb{C} : \sigma > \sigma_{1j}\}$, $\sigma_{1j} < \sigma_{aj}$, $j = 1, ..., n$, all poles being included in a compact set, and, for $\sigma > \sigma_{1j}$, the estimates

$$f_j(\sigma+it) \ll |t|^{\alpha_j}, \quad \alpha_j > 0, \quad |t| \geqslant t_0 > 0, \tag{8}$$

and

$$\int_{-T}^{T} |f_j(\sigma+it)|^2 \,\mathrm{d}\,t \ll T, \quad T \to \infty, \tag{9}$$

$j = 1, ..., r$, are satisfied. Moreover, we assume that $\lambda_{mj} = \lambda_m$, $j = 1, ..., r$, and

$$\lambda_m \geqslant c(\log m)^{\delta}, \quad c > 0, \quad \delta > 0. \tag{10}$$

Let $\mathbb{C}^r = \underbrace{\mathbb{C} \times ... \times \mathbb{C}}_{r}$. On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ define, for $\sigma_1 > \sigma_{11}, ..., \sigma_r > \sigma_{1r}$, an $\mathbb{C}^r$-valued random element $F = F(\sigma_1, ..., \sigma_r; \omega)$ by

$$F = F(\sigma_1, ..., \sigma_r, \omega) = (f_1(\sigma_1, \omega), ..., f_n(\sigma_r, \omega)),$$

where

$$f_j(\sigma_j, \omega) = \sum_{m=1}^{\infty} a_{mj}\omega(m)\mathrm{e}^{-\lambda_m \sigma_j}, \quad \omega \in \Omega.$$

Theorem B [13]. *Suppose that the system $\{\lambda_m\}$ is linearly independent over the field of rational numbers, and that conditions (8)-(10) are satisfied. Then the probability*

$$P_T(A) \stackrel{def}{=} \nu_T\big((f_1(\sigma_1 + it), ..., f_r(\sigma_r + it)) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}^r),$$

for $\sigma_1 > \sigma_{1j}, ..., \sigma_r > \sigma_{1r}$, converges weakly to the distribution of the random element $F(\sigma_1, ..., \sigma_r; \omega)$ as $T \to \infty$.

The aim of this note is to change condition (10) in Theorem B by a weaker one and to consider a general case of different exponents λ_{mj}. Therefore, for the proof we will apply a method different from that of [13]. Suppose that, for $\sigma_j > \sigma_{1j}$,

$$\sum_{m=1}^{\infty} |a_{mj}|^2 \mathrm{e}^{-2\lambda_{mj}\sigma_j} \log^2 m < \infty, \quad j = 1, ..., r. \tag{11}$$

Moreover, define $\Omega^r = \Omega_1 \times ... \times \Omega_r$ where $\Omega_j = \Omega$ for $j = 1, ..., r$. Then Ω^r is also a compact topological Abelian group. Denote by m_{Hr} the probability Haar measure on $(\Omega^r, \mathcal{B}(\Omega^r))$.

In the next section it will be proved that, under condition (11), for $\sigma_1 > \sigma_{11}, ..., \sigma_r > \sigma_{1r}$,

$$F(\sigma_1, ..., \sigma_r; \underline{\omega}) = (f_1(\sigma_1, \omega_1), ..., f_r(\sigma_r, \omega_r)),$$

where

$$f_j(\sigma_j, \omega_j) = \sum_{m=1}^{\infty} a_{mj}\omega_j(m)\mathrm{e}^{-\lambda_{mj}\sigma_j}, \, \omega_j \in \Omega_j, \quad j = 1, ..., r, \, \underline{\omega} = (\omega_1, ..., \omega_r),$$

is a $\mathbb{C}^n$-valued random element defined on the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_{Hr})$.

Theorem 1. *Suppose that the set $\bigcup_{j=1}^{r} \bigcup_{m=1}^{\infty} \{\lambda_{mj}\}$ is linearly independent over the field of rational numbers, and that conditions (8), (9) and (11)* *are satisfied. Then the probability measure P_T converges weakly to the distribution of the random element $F(\sigma_1, ..., \sigma_r; \underline{\omega})$ as $T \to \infty$.*

Note that joint limit theorems can be used to derive the joint universality for considered functions, see, for example, [16] and [17].

2. The random element $F(\sigma_1, ..., \sigma_r; \underline{\omega})$

In this section we will prove that, under condition (11), $F(\sigma_1, ..., \sigma_r; \underline{\omega})$ is a $\mathbb{C}^r$-valued random element. For the proof, we will apply a Rademacher's theorem on series of pairwise orthogonal random variables. Denote by $\mathbb{E}\xi$ the expectation of the random element ξ.

Lemma 2.[20] *Suppose that $\{X_n\}$ is a sequence of orthogonal random variables such that*

$$\sum_{m=1}^{\infty} \mathbb{E}|X_m|^2 \log^2 m < \infty.$$

Then the series

$$\sum_{m=1}^{\infty} X_m$$

converges almost surely.

Theorem 3. *Suppose that condition (11) holds. Then $F(\sigma_1, ..., \sigma_r; \underline{\omega})$, for*

$\sigma_1 > \sigma_{11}, ..., \sigma_r > \sigma_{1r}$, is a $\mathbb{C}^r$-valued random element defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_{Hr})$.

Proof. Clearly, it suffices to prove that, for each $j = 1, ..., r$,

$$f_j(\sigma_j, \omega) = \sum_{m=1}^{\infty} a_{mj} \omega(m) \mathrm{e}^{-\lambda_{mj}\sigma_j}, \quad \omega \in \Omega,$$

for $\sigma_j > \sigma_{1j}$, is a complex-valued random variable on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$.

We fix $j \in \{1, ..., r\}$. Let $\sigma > \sigma_{1j}$ be fixed, and

$$\xi_{mj} = \xi_{mj}(\omega) = a_{mj}\omega(m)\mathrm{e}^{-\lambda_{mj}\sigma}.$$

Then $\{\xi_{mj}\}$ is a sequence of pairwise orthogonal complex-valued random variables defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Really, denoting by $\overline{z}$ the complex conjugate of $z \in \mathbb{C}$, we find

$$\begin{aligned}\mathbb{E}(\xi_{mj}, \overline{\xi}_{kj}) &= \int_{\Omega} \xi_{mj}(\omega)\overline{\xi}_{kj}(\omega)\mathrm{d}m_H = a_{mj}\overline{a}_{kj}\mathrm{e}^{-(\lambda_{mj}+\lambda_{kj})\sigma} \int_{\Omega} \omega(m)\overline{\omega(k)}\mathrm{d}m_H \\ &= \begin{cases} 0 & \text{if } m \neq k, \\ |a_{mj}|^2 \mathrm{e}^{-2\lambda_{mj}\sigma} & \text{if } m = k. \end{cases}\end{aligned}$$

Since $\sigma > \sigma_{1j}$, hence we have in view of (11) that

$$\sum_{m=1}^{\infty} \mathbb{E}|\xi_{mj}|^2 \log^2 m < \infty.$$

This and Lemma 2 show that the series

$$\sum_{m=1}^{\infty} \xi_{mj} = \sum_{m=1}^{\infty} a_{mj}\omega(m)\mathrm{e}^{-\lambda_{mj}\sigma} = f(\sigma,\omega) \tag{12}$$

converges almost surely with respect the Haar measure m_H. Then

$$\Big(\sum_{m=1}^{\infty} a_{m1}\omega_1(m)\mathrm{e}^{-\lambda_{m1}\sigma_1}, ..., \sum_{m=1}^{\infty} a_{mr}\omega_r(m)\mathrm{e}^{-\lambda_{mr}\sigma_r} \Big)$$

converges almost surely in $\mathbb{C}^r$, and this proves the theorem. We note that $m_{Hr} = \underbrace{m_H \times ... \times m_H}_{r}$.

3. Joint limit theorems for Dirichlet polynomials

We start with a joint limit theorem on the torus Ω^r. Define the probability measure

$$Q_{T,r}(A) = \nu_T\Big(\big((\mathrm{e}^{it\lambda_{m1}} : m \in \mathbb{N}), ..., (\mathrm{e}^{it\lambda_{mr}} : m \in \mathbb{N})\big) \in A\Big).$$

Lemma 4. *The probability measure $Q_{T,r}$ converges weakly to the Haar measure m_{Hr} on $(\Omega^r, \mathcal{B}(\Omega^r))$ as $T \to \infty$.*

Proof. The dual group of Ω^r is

$$\bigoplus_{j=1}^{r} \bigoplus_{m=1}^{\infty} \mathbb{Z}_{mj},$$

where $\mathbb{Z}_{mj} = \mathbb{Z}$ for all $m \in \mathbb{N}$ and $j = 1, ..., r$.

$$(\underline{k}_1, ..., \underline{k}_r) = (k_{11}, k_{21}, ..., k_{1r}, k_{2r}, ...) \in \bigoplus_{j=1}^{r} \bigoplus_{m=1}^{\infty} \mathbb{Z}_{mj},$$

where only a finite number of integers k_{mj}, $m \in \mathbb{N}$, $j = 1, ..., r$, are distinct from zero, acts on Ω^r by

$$(\underline{x}_1, ..., \underline{x}_r) \to (\underline{x}_1^{\underline{k}_1}, ..., \underline{x}_r^{\underline{k}_r}) = \prod_{j=1}^{r} \prod_{m=1}^{\infty} x_{mj}^{k_{mj}}, \quad \underline{x}_j = (x_{1j}, x_{2j}, ...),\ x_{mj} \in \gamma,$$

$m \in \mathbb{N}$, $j = 1..., r$. Therefore, the Fourier transform $g_{T,r}(\underline{k}_1, ..., \underline{k}_r)$ of the measure $Q_{T,r}$ is

$$\begin{aligned} g_{T,r}(\underline{k}_1, ..., \underline{k}_r) &= \int\limits_{\Omega^r} \prod_{j=1}^{r} \prod_{m=1}^{\infty} x_{mj}^{k_{mj}} \, \mathrm{d}\, Q_{T,r} = \frac{1}{T} \int_0^T \prod_{j=1}^{r} \prod_{m=1}^{\infty} \mathrm{e}^{itk_{mj}\lambda_{mj}} \, \mathrm{d}\, t \\ &= \frac{1}{T} \int_0^T \exp\{it \sum_{j=1}^{r} \sum_{m=1}^{\infty} k_{mj}\lambda_{mj}\} \, \mathrm{d}\, t. \end{aligned}$$

Since the set $\bigcup_{j=1}^{r} \bigcup_{m=1}^{\infty} \{\lambda_{mj}\}$ is linearly independent over the field of rational numbers, hence we find that

$$g_{T,r}(\underline{k}_1, ..., \underline{k}_r) = \begin{cases} 1 & \text{if } (\underline{k}_1, ..., \underline{k}_r) = (\underline{0}, ..., \underline{0}), \\ \dfrac{\exp\left\{iT \sum\limits_{j=1}^{r} \sum\limits_{m=1}^{\infty} k_{mj}\lambda_{mj}\right\} - 1}{iT \sum\limits_{j=1}^{r} \sum\limits_{m=1}^{\infty} k_{mj}\lambda_{mj}} & \text{if } (\underline{k}_1, ..., \underline{k}_r) \neq (\underline{0}, ..., \underline{0}). \end{cases}$$

Therefore,

$$\lim_{T \to \infty} g_{T,r}(\underline{k}_1, ..., \underline{k}_r) = \begin{cases} 1 & \text{if } (\underline{k}_1, ..., \underline{k}_r) = (\underline{0}, ..., \underline{0}), \\ 0 & \text{if } (\underline{k}_1, ..., \underline{k}_r) \neq (\underline{0}, ..., \underline{0}). \end{cases}$$

This and continuity theorems for probability measures on compact groups [7] show that the probability measure $Q_{T,r}$ converges weakly to the Haar measure m_{Hr} as $T \to \infty$.

Let $\sigma_{2j} > \sigma_{aj} - \sigma_{1j}$, and, for $m, n \in \mathbb{N}$,

$$v_j(m, n) = \exp\{-\mathrm{e}^{(\lambda_m - \lambda_n)\sigma_{2j}}\}, \quad j = 1, ..., r.$$

Define, for $N_j \in \mathbb{N}$, $\sigma_j > \sigma_{1j}$ and $\widehat{\omega}_j \in \Omega$,

$$f_{N_j,j,n}(\sigma_j + it) = \sum_{m=1}^{N_j} a_{mj} v_j(m, n) \mathrm{e}^{-\lambda_{mj}(\sigma_j + it)},$$

$$f_{N_j,j,n}(\sigma_j + it, \widehat{\omega}_j) = \sum_{m=1}^{N_j} a_{mj} \widehat{\omega}_j(m) v_j(m, n) \mathrm{e}^{-\lambda_{mj}(\sigma_j + it)}, \quad j = 1, ..., r,$$

and consider the weak convergence of the probability measures

$$P_{T,N_1,...,N_r,n}(A) = \nu_T\big((f_{N_1,1,n}(\sigma_1 + it), ..., f_{N_r,r,n}(\sigma_r + it)\big) \in A$$

and

$$\widehat{P}_{T,N_1,...,N_r,n}(A) = \nu_T\big((f_{N_1,1,n}(\sigma_1 + it, \widehat{\omega}_1), ..., f_{N_r,r,n}(\sigma_r + it, \widehat{\omega}_r)\big) \in A,$$

where $(\widehat{\omega}_1, ..., \widehat{\omega}_r) \in \Omega^r$ and $A \in \mathcal{B}(\mathbb{C}^r)$.

Theorem 5. *The probability measures* $P_{T,N_1,...,N_r,n}$ *and* $\widehat{P}_{T,N_1,...,N_r,n}$ *both converge weakly to the same probability measure on* $(\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r))$ *as* $T \to \infty$.

Proof. Let a function $h : \Omega^r \to \mathbb{C}^r$ be given by

$$h(\omega_1, ..., \omega_r) = \Bigg(\sum_{m=1}^{N_1} a_{m1} v(m,n) \mathrm{e}^{-\lambda_{m1}\sigma_1} \omega_1^{-1}(m), ..., \sum_{m=1}^{N_r} a_{mr} v(m,n) \mathrm{e}^{-\lambda_{mr}\sigma_r} \omega_r^{-1}(m) \Bigg),$$

$(\omega_1, ..., \omega_r) \in \Omega^r$. Then, clearly,

$$h\Bigg((\mathrm{e}^{it\lambda_{m1}} : m \in \mathbb{N}), ..., (\mathrm{e}^{it\lambda_{mr}} : m \in \mathbb{N}) \Bigg) = \big(f_{N_1,1,n}(\sigma_1 + it), ..., f_{N_r,r,n}(\sigma_r + it) \big) \stackrel{def}{=} f_{N_1,...,N_r,n}(\sigma_1, ..., \sigma_r; t),$$

and the function h is continuous. Therefore, $P_{T,N_1,...,N_r,n} = Q_{T,r} h^{-1}$, and by Theorem 5.1 of [1] and Lemma 4 the probability measure $P_{T,N_1,...,N_r,n}$ converges weakly to $m_{Hr} h^{-1}$ as $T \to \infty$.

Now let $h_1 : \Omega^r \to \Omega^r$ be defined by the formula

$$h_1(\omega_1, ..., \omega_r) = (\omega_1 \widehat{\omega}_1^{-1}, ..., \omega_r \widehat{\omega}_r^{-1}).$$

Then we have that

$$\big(f_{N_1,1,n}(\sigma_1 + it, \widehat{\omega}_1), ..., f_{N_r,1,n}(\sigma_r + it, \widehat{\omega}_r) \big) = h\big(h_1\big((\mathrm{e}^{it\lambda_{m_1}} : m \in \mathbb{N}), ..., (\mathrm{e}^{it\lambda_{m_r}} : m \in \mathbb{N}) \big) \big).$$

Similarly to the case of the measure $P_{T,N_1,...,N_r,n}$ we obtain that the probability measure $P_{T,N_1,...,N_r,n}$ converges weakly to the measure $m_{Hr}(hh_1)^{-1}$ as $T \to \infty$. The Haar measure m_{Hr} is invariant with respect to translations by points from Ω^r. Therefore,

$$m_{Hr}(hh_1)^{-1} = (m_{Hr} h_1^{-1}) h^{-1} = m_{Hr} h^{-1},$$

and the theorem is proved.

4. Limit theorems for absolutely convergent series

Define, for $\omega_j \in \Omega$ and $j = 1, ..., r$,

$$f_{n,j}(s) = \sum_{m=1}^{\infty} a_{mj} v_j(m,n) \mathrm{e}^{-\lambda_{mj} s}$$

and

$$f_{n,j}(s, \omega_j) = \sum_{m=1}^{\infty} a_{mj} \omega_j(m) v_j(m,n) \mathrm{e}^{-\lambda_{mj} s}.$$

Then the latter series both converge absolutely for $\sigma > \sigma_{1j}$. The proof of this is given in [12], Lemma 4. In this section we consider the weak convergence of the probability measures

$$P_{T,n}(A) = \nu_T\big(((f_{n,1}(\sigma_1 + it), ..., f_{n,r}(\sigma_r + it)) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}^r),$$

and

$$\widehat{P}_{T,n}(A) = \nu_T\big(((f_{n,1}(\sigma_1 + it, \omega_1), ..., f_{n,r}(\sigma_r + it, \omega_r)) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}^r).$$

Theorem 6. *Let $\sigma_j > \sigma_{1j}$, $j = 1, ..., r$. Then there exists a probability measure P_n on $(\mathbb{C}^r, \mathcal{B}(\mathbb{C}))$ such that the measures $P_{T,n}$ and $\widehat{P}_{T,n}$ both converge weakly to P_n as $T \to \infty$.*

Proof. We will apply Theorem 5. Without loss of generality we take $N_1 = ... = N_r \stackrel{def}{=} N$. Then by Theorem 5 the measures $P_{T,N_1,...,N_r,n} \stackrel{def}{=} P_{T,N,n}$ and $\widehat{P}_{T,N_1,...,N_r,n} \stackrel{def}{=} \widehat{P}_{T,N,n}$ both converge weakly to the same measure $P_{N,n}$, say, as $T \to \infty$.

First we will prove that the family of probability measures $\{P_{N,n}\}$ is tight for fixed n. Let η be a random variable defined on a certain probability space $(\widehat{\Omega}, \mathcal{F}, \mathbb{P})$ and uniformly distributed on $[0, 1]$, and let, for $j = 1, ..., r$,

$$X_{T,N,j,n} = X_{T,N,j,n}(\sigma_j) = f_{N,j,n}(\sigma_j + iT\eta).$$

Then we have that

$$\underline{X}_{T,N,n} \stackrel{def}{=} \big(X_{T,N,1,n}, ..., X_{T,N,r,n}\big) \xrightarrow[T\to\infty]{\mathcal{D}} \underline{X}_{N,n}, \tag{12}$$

where $\underline{X}_{N,n}$ is a $\mathbb{C}^r$-valued random element with distribution $P_{N,n}$, and $\xrightarrow{\mathcal{D}}$ means the convergence in distribution.

Let $\underline{z}_1 = (z_{11}, ..., z_{1r})$, $\underline{z}_2 = (z_{21}, ..., z_{2r}) \in \mathbb{C}^r$. Define a metric ρ in $\mathbb{C}^r$ by

$$\rho(\underline{z}_1,\underline{z}_2)=\Big(\sum_{j=1}^{r}|z_{1j}-z_{2j}|^2\Big)^{\frac{1}{2}}.$$

Then, clearly, this metric induces the topology of $\mathbb{C}^r$.

Since the series for $f_{n,j}$ converges absolutely for $\sigma>\sigma_{1j}$, $j=1,...,r$, we obtain, for $M>0$,

$$\begin{aligned}
&\limsup_{T\to\infty}\mathbb{P}\big(\rho(\underline{X}_{T,N,n},\underline{0})>M\big)\leqslant\\
&\qquad\leqslant\frac{1}{M}\sup_{N\geqslant 1}\limsup_{T\to\infty}\frac{1}{T}\int\limits_0^T\rho\big(\underline{f}_{N,n}(\sigma_1,...,\sigma_r;t),\underline{0}\big)\,\mathrm{d}\,t=\\
&\qquad=\frac{1}{M}\sup_{N\geqslant 1}\limsup_{T\to\infty}\frac{1}{T}\int_0^T\left(\sum_{j=1}^{r}|f_{N,j,n}(\sigma_j+it)|^2\right)^{\frac{1}{2}}\mathrm{d}\,t\leqslant\\
&\qquad\leqslant\frac{1}{M}\sup_{N\geqslant 1}\limsup_{T\to\infty}\left(\frac{1}{T}\sum_{j=1}^{r}\int_0^T|f_{N,j,n}(\sigma_j+it)|^2\,\mathrm{d}\,t\right)^{\frac{1}{2}}=\\
&\qquad=\frac{1}{M}\sup_{N\geqslant 1}\Big(\sum_{j=1}^{r}\sum_{m=1}^{N}|a_{mj}|^2v_j^2(m,n)\mathrm{e}^{-2\lambda_{mj}\sigma_j}\Big)^{\frac{1}{2}}\leqslant R<\infty,
\end{aligned}\tag{13}$$

where

$$\underline{f}_{N,n}(\sigma_1,...,\sigma_r;t)=\big(f_{N,1,n}(\sigma_1+it),...,f_{N,r,n}(\sigma_r+it)\big).$$

Now we take $M=R\epsilon^{-1}$, where ϵ is an arbitrary positive number. Then (13) yields

$$\limsup_{T\to\infty}\mathbb{P}\big(\rho(\underline{X}_{T,N,n},\underline{0})>M\big)\leqslant\varepsilon.$$

This and (12) imply the inequality

$$\mathbb{P}\big(\rho(\underline{X}_{T,N,n},\underline{0})>M\big)\leqslant\varepsilon.\tag{14}$$

Now we define

$$K_\epsilon=\{\underline{z}\in\mathbb{C}^r:\ \rho(\underline{z},\underline{0})\leqslant M\}.$$

Then, obviously, K_ϵ is a compact subset of the space $\mathbb{C}^r$. In view of (14) and of the definition of $P_{N,n}$

$$P_{N,n}(K_\epsilon)\geqslant 1-\epsilon$$

for all $N \in \mathbb{N}$. This shows that the tightness of the family $\{P_{N,n}\}$. Hence, by the Prokhorov theorem, see, for example, [1], the latter family is relatively compact.

By the definition of $f_{n,j}(s)$ and $f_{N,n,j}(s)$, for $\sigma > \sigma_{1j}$,

$$\lim_{N\to\infty} f_{N,j,n}(s) = f_{n,j}(s), \quad j = 1, ..., r,$$

and the series for $f_{n,j}(s)$ absolutely converges. Therefore, denoting

$$\underline{f}_n(\sigma_1, ..., \sigma_r; t) = \big(f_{n,1}(\sigma_1 + it), ..., f_{n,r}(\sigma_r + it)\big),$$

we have, for every $\epsilon > 0$ and $\sigma_j > \sigma_{1j}$, $j = 1, ..., r$, that

$$\lim_{N\to\infty} \limsup_{T\to\infty} \nu\big(\rho(\underline{f}_{N,n}(\sigma_1, ..., \sigma_r; t), \underline{f}_n(\sigma_1, ..., \sigma_r; t)) \geqslant \epsilon\big) \leqslant$$
$$\leqslant \lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{\epsilon T} \int_0^T \rho(\underline{f}_{N,n}(\sigma_1, ..., \sigma_r; t), \underline{f}_n(\sigma_1, ..., \sigma_r; t)) \, \mathrm{d}\, t = 0. \quad (15)$$

Define, for $\sigma_j > \sigma_{1j}$,

$$X_{T,j,n} = X_{T,n}(\sigma_j) = f_{n,j}(\sigma_j + iT\eta), \quad j = 1, ..., r,$$

and put

$$\underline{X}_{T,n} = \big(X_{T,1,n}, ..., X_{T,r,n}\big).$$

Then by (15)

$$\lim_{N\to\infty} \limsup_{T\to\infty} \mathbb{P}\big(\rho(\underline{X}_{T,N,n}, \underline{X}_{T,n}) \geqslant \epsilon\big) = 0. \quad (16)$$

The family $\{P_{N,n}\}$ is relatively compact. Therefore, there exists a subsequence $\{P_{N',n}\} \subset \{P_{N,n}\}$ which converges weakly to the probability measure P_n, say, as $N' \to \infty$. Then

$$\underline{X}_{N',n} \xrightarrow[N'\to\infty]{\mathcal{D}} P_n. \quad (17)$$

The space $\mathbb{C}^r$ is separable. Therefore, (12), (16) and (17) show that the conditions of Theorem 4.2 from [1] are satisfied. Consequently,

$$\underline{X}_{T,n} \xrightarrow[T\to\infty]{\mathcal{D}} P_n, \quad (18)$$

i.e. the measure $P_{T,n}$ converges weakly to the probability measure P_n on $(\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r))$ as $T \to \infty$.

In view of (18), the measure P_n is independent of the subsequence $\{P_{N',n}\}$. Therefore, by (17)

$$\underline{X}_{N,n} \xrightarrow[N\to\infty]{\mathcal{D}} P_n. \quad (19)$$

Now, repeating the above arguments for the random elements

$$\underline{\widehat{X}}_{T,N,n} = \big(\widehat{X}_{T,N,1,n}, ..., \widehat{X}_{T,N,r,n}\big)$$

and

$$\underline{\widehat{X}}_{T,n} = \big(\widehat{X}_{T,1,n}, ..., \widehat{X}_{T,r,n}\big),$$

where

$$\widehat{X}_{T,N,j,n} = \widehat{X}_{T,N,j,n}(\sigma_j, \omega_j) = f_{N,j,n}(\sigma_j + iT\eta, \omega_j), \quad j = 1, ..., r,$$

$$\widehat{X}_{T,j,n} = \widehat{X}_{T,j,n}(\sigma_j, \omega_j) = f_{j,n}(\sigma_j + iT\eta, \omega_j), \quad j = 1, ..., r,$$

and taking into account (19), we obtain that the probability measure $\widehat{P}_{T,n}$ also converges weakly to P_n as $T \to \infty$. The theorem is proved.

5. Approximation in the mean

To pass from the functions $f_{n,j}(s)$ to $f_j(s)$ we need an approximation in the mean of $f_1(s), ..., f_r(s)$ and of $f_1(s, \omega_1), ..., f_r(s, \omega_r)$ by $f_{n,1}(s), ..., f_{n,r}(s)$ and by $f_{n,1}(s, \omega_1), ...,$
$f_{n,r}(s, \omega_r)$, respectively. Let

$$\underline{f}(\sigma_1, ..., \sigma_r; t) = \big(f_1(\sigma_1 + it), ..., f_r(\sigma_r + it)\big),$$

and

$$\underline{f}(\sigma_1, ..., \sigma_r; t, \underline{\omega}) = \big(f_1(\sigma_1 + it, \omega_1), ..., f_r(\sigma_r + it, \omega_r)\big),$$

$$\underline{f}_n(\sigma_1, ..., \sigma_r; t, \underline{\omega}) = \big(f_{n,1}(\sigma_1 + it, \omega_1), ..., f_{n,r}(\sigma_r + it, \omega_r)\big).$$

Theorem 7. *Let $\sigma_j > \sigma_{1j}$, $j = 1, ..., r$. Then*

$$\lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \rho(\underline{f}(\sigma_1, ..., \sigma_r; t), \underline{f}_n(\sigma_1, ..., \sigma_r; t)) \,\mathrm{d}\, t = 0$$

and

$$\lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \rho(\underline{f}(\sigma_1, ..., \sigma_r; t, \underline{\omega}), \underline{f}_n(\sigma_1, ..., \sigma_r; t, \underline{\omega})) \,\mathrm{d}\, t = 0$$

for almost all $(\omega_1, ..., \omega_r)$.

Proof. Suppose that the function $f(s)$ satisfies the conditions of Theorem A, and for $\sigma > \sigma_1$,

$$f_n(s) = \sum_{m=1}^{\infty} a_m v(m, n) \mathrm{e}^{-\lambda_m s},$$

$$f_n(s,\omega)=\sum_{m=1}^{\infty}a_m\omega(m)v(m,n)\mathrm{e}^{-\lambda_m s},$$

where $v(m,n)=\exp\{-\mathrm{e}^{-(\lambda_n-\lambda_m)\sigma_2}\}$ with $\sigma_2>\sigma_a-\sigma_1$, and $\omega\in\Omega$. Then in [12] it was obtained that, for $\sigma>\sigma_1$,

$$\lim_{N\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T|f(\sigma+it)-f_n(\sigma+it)|\,\mathrm{d}\,t=0$$

and

$$\lim_{N\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T|f(\sigma+it,\omega)-f_n(\sigma+it,\omega)|\,\mathrm{d}\,t=0$$

for almost all $\omega\in\Omega$. Since

$$\rho(\underline{z}_1,\underline{z}_2)\leqslant\sum_{j=1}^{r}|z_{1j}-z_{2j}|,$$

hence the theorem follows.

6. Joint limit theorems for $f_j(s)$ and $f_j(s,\omega)$

In this section we begin to prove Theorem 1. We will prove limit theorems for the vectors $\underline{f}(\sigma_1,...,\sigma_r;t)$ and $\underline{f}(\sigma_1,...,\sigma_r;t,\underline{\omega})$ defined in Section 5.

Theorem 8. *Let $\sigma_j>\sigma_{1j}$, $j=1,...,r$. Then the probability measures P_T and*

$$\widehat{P}_T(A)=\nu_T\big(\underline{f}(\sigma_1,...,\sigma_r;t,\underline{\omega})\in A\big),\quad A\in\mathcal{B}(\mathbb{C}^r),$$

both converge weakly to the same probability measure on $(\mathbb{C}^r,\mathcal{B}(\mathbb{C}^r))$ as $T\to\infty$.

Proof. We argue similarly to the proof of Theorem 6. By Theorem 6 the probability measures $P_{T,n}$ and $\widehat{P}_{T,n}$ converge weakly to the same measure P_n on $(\mathbb{C}^r,\mathcal{B}(\mathbb{C}^r))$ as $T\to\infty$. We will show that the family of probability measures $\{P_n: n\in\mathbb{N}\}$ is tight. For this, we will preserve the notation of previous sections.

By Theorem 6

$$\underline{X}_{T,n}\xrightarrow[T\to\infty]{\mathcal{D}}\underline{X}_n,\tag{20}$$

where $\underline{X}_n$ is a $\mathbb{C}^r$-valued random element with distribution P_n. Since the series (11) converges and the series for each $f_{n,j}$ converges absolutely, we have, for $M > 0$,

$$\limsup_{T\to\infty} \mathbb{P}\big(\rho(\underline{X}_{T,n}, \underline{0}) > M\big) \leqslant$$

$$\leqslant \frac{1}{M} \sup_{n\geqslant 1} \limsup_{T\to\infty} \frac{1}{T} \int\limits_0^T \rho\big(\underline{f}_n(\sigma_1, ..., \sigma_r; t), \underline{0}\big) \,\mathrm{d}\, t =$$

$$= \frac{1}{M} \sup_{n\geqslant 1} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \left(\sum_{j=1}^r |f_{n,j}(\sigma_j + it)|^2 \right)^{\frac{1}{2}} \mathrm{d}\, t \leqslant$$

$$\leqslant \frac{1}{M} \sup_{n\geqslant 1} \limsup_{T\to\infty} \left(\frac{1}{T} \sum_{j=1}^r \int_0^T |f_{n,j}(\sigma_j + it)|^2 \,\mathrm{d}\, t \right)^{\frac{1}{2}} =$$

$$= \frac{1}{M} \sup_{n\geqslant 1} \left(\sum_{j=1}^r \sum_{m=1}^\infty |a_{mj}|^2 v_j^2(m,n) \mathrm{e}^{-2\lambda_{mj}\sigma_j} \right)^{\frac{1}{2}} \leqslant$$

$$\leqslant \frac{1}{M} \left(\sum_{j=1}^r \sum_{m=1}^\infty |a_{mj}|^2 \mathrm{e}^{-2\lambda_{mj}\sigma_j} \right)^{\frac{1}{2}} \leqslant R < \infty.$$

Hence, taking $M = R\epsilon^{-1}$, we find that

$$\limsup_{T\to\infty} \mathbb{P}\big(\rho(\underline{X}_{T,n}, \underline{0}) > M\big) \leqslant \epsilon.$$

Consequently, in view of (20)

$$\mathbb{P}\big(\rho(\underline{X}_n, \underline{0}) > M\big) \leqslant \epsilon.$$

This shows that

$$P_n(K_\epsilon) \geqslant 1 - \epsilon$$

for all $n \in \mathbb{N}$, i.e. the family $\{P_n\}$ is tight. Hence, by the Prokhorov theorem, it is relatively compact. Therefore, there exists a subsequence $\{P_{n_1}\} \subset \{P_n\}$ which converges weakly to the probability measure P, say, on $(\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r))$ as $n_1 \to \infty$. Then

$$\underline{X}_{n_1} \xrightarrow[n_1\to\infty]{\mathcal{D}} P. \tag{21}$$

Let, for $\sigma_j > \sigma_{1j}$

$$X_{T,j} = X_{T,j}(\sigma_j) = f_j(\sigma_j + iT\eta), \quad j = 1, ..., r,$$

and

$$\underline{X}_T = \big(X_{T,1}, ..., X_{T,r}\big).$$

Then by the first assertion of Theorem 7

$$\lim_{n\to\infty} \limsup_{T\to\infty} \mathbb{P}\big(\rho(\underline{X}_{T,n}, \underline{X}_T) \geqslant \epsilon\big) \leqslant$$

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{\epsilon T} \int_0^T \rho\big(\underline{f}_n(\sigma_1, ..., \sigma_r; t), \underline{f}(\sigma_1, ..., \sigma_r; t)\big) = 0.$$

This, (20), (21) and Theorem 4.2 of [1] show that

$$\underline{X}_T \xrightarrow[T\to\infty]{\mathcal{D}} P. \tag{23}$$

Now let, for $\sigma_j > \sigma_{1j}$,

$$\widehat{X}_{T,j} = \widehat{X}_{T,j}(\sigma_j) = f_j(\sigma_j + iT\eta, \omega_j), \quad j = 1, ..., r,$$

and

$$\underline{\widehat{X}}_T = (\widehat{X}_{T,1}, ..., \widehat{X}_{T,r}).$$

Then, reasoning similarly above for the vectors $\widehat{X}_{T,n}$ and $\underline{\widehat{X}}_T$, and using (23) and the second assertion of Theorem 7, we obtain that the probability measure $\widehat{P}_T$ also converges to P as $T \to \infty$. The theorem is proved.

7. Proof of Theorem 1

It remains to identify the limit measure P in Theorem 8. For this, we will apply some elements of the ergodic theory.

Let $a_{t,j} = \{e^{-i\lambda_{mj}t} : m \in \mathbb{N}\}$ for $t \in \mathbb{R}$, $j = 1, ..., r$. Then, for each j, $\{a_{t,j} : t \in \mathbb{R}\}$ is a one–parameter group. We define the one–parameter family $\{\varphi_{t,j} : t \in \mathbb{R}\}$ of transformations on Ω_j by $\varphi_{t,j} = a_{t,j}\omega_j$ for $\omega_j \in \Omega_j$, $j = 1, ..., r$. Then we obtain a one parameter group $\{\varphi_{t,j} : t \in \mathbb{R}\}$ of measurable transformations on Ω_j, $j = 1, ..., r$.

Define $\{\Phi_t : t \in \mathbb{R}\} = \{\varphi_{t,1} : t \in \mathbb{R}\} \times ... \times \{\varphi_{t,r} : t \in \mathbb{R}\}$. Then $\{\Phi_t : t \in \mathbb{R}\}$ is a one-parameter group of measurable transformations on Ω^r.

Lemma 9. *The one-parameter group* $\{\Phi_t : t \in \mathbb{R}\}$ *is ergodic.*

Proof. In [18] it was proved that $\{\varphi_{t,j} : t \in \mathbb{R}\}$ for each $j = 1, ..., r$ is an ergodic one–parameter group. Hence the lemma follows.

Proof of Theorem 1. Let $A \in \mathcal{B}(\mathbb{C}^r)$ be a continuity set of the measure P in Theorem 8. Then, by Theorem 10, for $\sigma_1 > \sigma_{11}, ..., \sigma_r > \sigma_{1r}$,

$$\lim_{T\to\infty} \nu_T\big(\underline{f}(\sigma_1, ..., \sigma_r; t, \underline{\omega}) \in A\big) = P(A) \tag{24}$$

for almost all $\underline{\omega} \in \Omega^r$. Now we fix the set A and define a random variable θ on $(\Omega^r, \mathcal{B}(\Omega^r), m_{Hr})$ by

$$\theta(\underline{\omega}) = \begin{cases} 1 & \text{if } F(\sigma_1, ..., \sigma_r; \underline{\omega}) \in A, \\ 0 & \text{if } \underline{F}(\sigma_1, ..., \sigma_r; \underline{\omega}) \notin A. \end{cases}$$

Then

$$\mathbb{E}(\theta) = \int_{\Omega_r} \theta \mathrm{d} m_{Hr} = m_{Hr}\big(\omega \in \Omega : \underline{F}(\sigma_1, ..., \sigma_r; \underline{\omega}) \in A\big) \overset{def}{=} P_F$$

is the distribution of the random element $\underline{F}$. Since by Lemma 9 the one–parameter group $\{\Phi_t : t \in \mathbb{R}\}$ is ergodic, the random process $\theta(\Phi_t(\underline{\omega}))$ is also ergodic. Therefore, by the Birkkhoff-Khinchine theorem

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \theta(\Phi_t(\underline{\omega}))\,\mathrm{d}\,t = \mathbb{E}(\theta) \tag{26}$$

for almost all $\underline{\omega} \in \Omega^r$. The definitions of θ and of $\{\Phi_t : t \in \mathbb{R}\}$ yield

$$\frac{1}{T} \int_0^T \theta(\Phi_t(\underline{\omega}))\,\mathrm{d}\,t = \nu_T\big(\underline{f}(\sigma_1, ..., \sigma_r; t, \underline{\omega}) \in A\big).$$

Hence and from (25), (26), we deduce that

$$\lim_{T\to\infty} \nu_T\big(\underline{f}(\sigma_1, ..., \sigma_r; t, \underline{\omega})\big) = P_F(A)$$

for almost all $\underline{\omega} \in \Omega^r$. Consequently, by (24)

$$P(A) = P_F(A)$$

for any continuity set A of the measure P. It is well known that all continuity sets constitute the determining class. Therefore,

$$P(A) = P_F(A)$$

for all $A \in \mathcal{B}(\mathbb{C}^r)$, and the theorem is proved.

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