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The generalized path algebras over standardly stratified algebras

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Abstract. In this note, it is proved that the generalized path algebras over standardly stratified algebras are also standardly stratified, and the generalized path algebras over quasi-hereditary algebras are also quasi-hereditary. The $\Delta-$good module categories over these big quasi-hereditary algebras are determined in terms of those of the given algebras.

Quasi-hereditary algebras and their generalizations such as standardly stratified algebras were first introduced by Cline, Parshall and Scott [2] [3] in order to study highest weight categories in the representation theory of semi-simple Lie algebras and algebraic groups. Since then, these algebras have been studied by many authors (for example: Dlab-Ringel [5] [6], Ringel [11], and so on). Many algebras which arise rather naturally have been shown to be quasi-hereditary: the Schur algebras [10], the Auslander algebras [6], and the endomorphism algebras of direct sum of all indecomposable Δ-good modules for any $\mathcal{F}(\Delta)$-finite quasi-hereditary algebra [12], the smash product of a quasi-hereditary graded algebra graded by a finite group [13] [14]. In this note, we will prove that the generalized path algebras [4] over quasi-hereditary algebras are quasi-hereditary. In fact we first prove a more general result: the generalized path algebras over standardly stratified algebras are standardly stratified; and then as a corollary, we prove the mentioned result above for quasi-hereditary algebras. Given a quasi-hereditary algebra (A, Λ),

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of central importance are the modules filtered by (co-)standard modules (the precise meaning will be given later on) and characteristic tilting modules [11]. We will give the precise description of $\Delta-$good module category of the quasi-hereditary generalized path algebras. It follows that for any finitely many quasi-hereditary algebras A_i with $\Delta-$good modules categories $\mathcal{F}_{A_i}(\Delta)$, $i = 1, 2, \cdots, n$, we can construct a big algebra via generalized path algebras such that roughly speaking the extension of their $\Delta-$good module categories (the precise meaning will be given later) is the $\Delta-$good module category of the big algebra.

Throughout the paper, K will denote a fixed field. By an algebra A, we mean an associative finite dimensional K-algebra. By a module M, we mean finitely generated left A-module. Now we recall the notation of the generalized path algebras from [4][8]. Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a quiver with Γ_0 the set of vertices and Γ_1 the set of arrows. An arrow α from $s(\alpha)$ to $e(\alpha)$ is sometimes denoted by $s(\alpha) \xrightarrow{\alpha} e(\alpha)$, $s(\alpha)$(or $e(\alpha)$) is called the starting vertex (ending vertex, resp.) of α. A *path* in Γ is $(b|\alpha_t \cdots \alpha_1|a)$, where $\alpha_i \in \Gamma_1$, for $i = 1, \cdots, t$, and $s(\alpha_1) = a$, $e(\alpha_i) = s(\alpha_{i+1})$ for $i = 1, \cdots, t-1$, and $e(\alpha_t) = b$, the *length of a path* is the number of arrows in it. To each arrow α we can assign an edge $\overline{\alpha}$ where the orientation is forgotten. A *walk* between two vertices a and b is given by $(b|\overline{\alpha_t} \cdots \overline{\alpha_1}|a)$, where the $a \in \{s(\alpha_1),\, e(\alpha_1)\}$, $b \in \{s(\alpha_n),\, e(\alpha_n)\}$, and for each $i = 1, \cdots, t-1$,

$$\{s(\alpha_i), e(\alpha_i)\} \cap \{s(\alpha_{i+1}), e(\alpha_{i+1})\} \neq \emptyset.$$

A quiver is said to be *connected* if for each pair of vertices a and b , there exists a walk between them.

Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a quiver and $\mathcal{A} = \{A_i | i \in \Gamma_0\}$ be a family of $K-$algebras A_i with identity, indexed by the vertices of Γ. Unless otherwise stated, we shall indicate the identity of A_i as e_i , for $i \in \Gamma_0$. The elements of $\bigcup_{i\in\Gamma_0} A_i$ are called the $\mathcal{A}-$*paths of length zero*, and for each $t \geq 1$, an $\mathcal{A} -$ *path of length t* is given by $a_{t+1}\beta_t a_t \cdots \beta_2 a_2 \beta_1 a_1$, where $(e(\beta_t)|\beta_t, \cdots, \beta_1|s(\beta_1))$ is a path in Γ of length t, for each $i = 1, \cdots, t$, $a_i \in A_{s(\beta_i)}$, and $a_{t+1} \in A_{e(\beta_t)}$, Consider now the quotient Λ of the K-vector space with basis the set of all $\mathcal{A}-$paths of Γ by the subspace generated by all the elements of the form

$$(a_{t+1}\beta_t \cdots a_{j+1}\beta_j(a_j^1 + \cdots + a_j^m)\beta_{j-1} \cdots \beta_1 a_1)$$

$$-\sum_{l=1}^{m}(a_{t+1}\beta_t \cdots a_{j+1}\beta_j a_j^l \beta_{j-1} \cdots \beta_1 a_1)$$

where $(e(\beta_t)|\beta_t \cdots \beta_1|s(\beta_1))$ is a path in Γ of length t , for each $i = 1, \cdots, t$, $a_i \in A_{s(\beta_i)}$, $a_{t+1} \in A_{e(\beta_t)}$, and $a_j^l \in A_{s(\beta_j)}$ for $l = 1, \cdots, m$.

Define now in Λ the following multiplication. Given two elements

$$[a_{t+1}\beta_t\cdots\beta_1 a_1],\ [b_{m+1}\gamma_m\cdots\gamma_1 b_1]$$

we define

$$[a_{t+1}\beta_t\cdots\beta_1 a_1][b_{m+1}\gamma_m\cdots\gamma_1 b_1] =$$
$$= \begin{cases} [a_{t+1}\beta_t\cdots\beta_1 a_1 b_{m+1}\gamma_m\cdots\gamma_1 b_1], & \text{if } b_{m+1},\ a_1 \text{belong} \\ & \text{to the same } A_i \\ 0, & \text{otherwise} \end{cases}$$

It is easy to check that the above multiplication in Λ is well-defined and gives Λ the structure of a $K-$algebra. The algebra Λ defined above is called $\mathcal{A}-$path algebra of Γ and we denote it by $\Lambda = K(\Gamma, \mathcal{A})$. Λ is also called *generalized path algebra.*

Remark 1. $\Lambda = K(\Gamma, \mathcal{A})$ has identity if and only if Γ_0 is finite. Moreover if $\Gamma_0 = \{1, \cdots, n\}$, then $e = e_1 + \cdots + e_n$ is identity of Λ. In the following, we always assume $\Lambda = K(\Gamma, \mathcal{A})$ has identity.

Remark 2. The usual path algebra K can be embedded into the $\mathcal{A}-$path algebra $K(\Gamma, \mathcal{A})$, or say if $A_i = K$, for each $i \in \Gamma_0$, then $K(\Gamma, \mathcal{A}) = K\Gamma$.

Remark 3. The generalized path algebra $\Lambda = K(\Gamma, \mathcal{A})$ is of finite dimension over K if and only if $\dim K A_i = \infty$ for each $i \in \Gamma_0$, and Γ is a finite quiver without oriented cycles [4]. In the following, we always assume $\Lambda = K(\Gamma, \mathcal{A})$ is finite dimensional.

We can give an alternative definition for the generalized path algebra $\Lambda = K(\Gamma, \mathcal{A})$ as follows. Let ${}_jM_i$ be the free $A_j - A_i-$bimodule with free generators given by the arrows from i to j. If $A = \bigoplus_{i\in\Gamma_0} A_i$, the ${}_jM_i$ is also an $A-A-$bimodule by defining $A_k \cdot {}_jM_i = 0$, if $k \neq j$ and ${}_jM_i \cdot A_k = 0$ if $k \neq i$. Let $M = \bigoplus_{i\to j} {}_jM_i$, which is clearly an $A-A-$bimodule. It is easy to check that Λ is isomorphic to the algebra

$$A \oplus M \oplus (M \otimes_K M) \oplus (M \otimes_K M \otimes_K M) \oplus \cdots$$

with multiplication given by the tensor product.

We now introduce the definition of quasi-hereditary algebras and standardly stratified algebras [2][3][7][10]. Quasi-hereditary algebras and standardly stratified algebras depend heavily on an ordering of the simple modules. For a finite dimensional algebra Λ over K, We fix an ordering on the simple $\Lambda-$modules: $E(1), E(2), \cdots, E(n)$. Let $P(i)$ be the projective cover of $E(i)$. We denote by $\Delta(i)$ the maximal factor of $P(i)$ with composition factors of the form $E(j)$, where $j = i$. Let

$\Delta = \{\Delta(1), \cdots, \Delta(n)\}$. We denote by $\mathcal{F}(\Theta)$ the full subcategory of ModΛ consisting of modules which have a filtration with factors in Θ, where ModΛ denotes the category of $f.g.$ left modules over Λ and Θ is a set of modules. These modules are said to be Θ-good. The algebra (Λ, E) is called *standardly stratified* with respect to the ordering of simple modules if $P(i) \in \mathcal{F}(\Delta)$, for all $i = 1, \cdots, n$ [7]; if in addition, End$\Lambda(\Delta(i))$ is a division ring, for all $i = 1, \cdots, n$, then (Λ, E) is *quasi-hereditary* [11].Standardly stratified algebras as a generalization of quasi-hereditary algebras have been studied recently by some authors in various aspects [1] [7] [9] [15].

Before stating our main result, we need to fix an ordering on the set of simple modules over the generalized path algebra. Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a finite connected quiver without oriented cycles. Let $\mathcal{A} = \{A_i | i \in \Gamma_0\}$ be a family of quasi-hereditary algebras A_i w.r.t. the ordering on the set E_i of simple A_i-modules. Since Γ is a finite quiver and connected and without oriented cycles, we can assume that $\Gamma_0 = \{1, 2, \cdots, n\}$, such that i is a source of the quiver obtained from Γ by deleting the vertices $1, 2, \cdots, i-1$. For any $i = 1, 2, \cdots, n$. Let $E_i = (E_i(1), \cdots, E_i(s_i))$ be the ordering on simple A_i-modules. Thus there is a complete set of orthogonal primitive idempotents $\underline{e}_i = (e_{i1}, \cdots, e_{is_i})$ of A_i corresponding to the ordered index set E_i of simple A_i-modules. Let $P_i(j) = A_i e_{ij}$. Then $P_i = (P_i(1), \cdots, P_i(S_i))$ is the corresponding set of indecomposable projective A_i-modules, and $\frac{P_i(j)}{radP_i(j)} \cong E_i(j)$, $i = 1, 2, \cdots, n$; $j = 1, 2, \cdots, s_i$. By identifying A_i with the subalgebra of $\Lambda = K(\Gamma, \mathcal{A})$ generated by paths of length 0 at the vertex i, $\underline{e} = (e_{11}, \cdots, e_{1s_1}, \cdots, e_{n1}, \cdots, e_{ns_n})$ is a complete set orthogonal primitive idempotents of $\Lambda = K(\Gamma, \mathcal{A})$. This index set is endowed with the ordering :$(i, j) < (k, h) \Leftrightarrow i < k$, or $i = k$ and $j < h$. With this notation , we have the following:

Theorem 1. Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a finite connected quiver without oriented cycles. Let $\mathcal{A} = \{A_i | i \in \Gamma_0\}$ be a family of standardly stratified algebras (A_i, E_i), $i \in \Gamma_0$. Then $\Lambda = K(\Gamma, \mathcal{A})$ is a standardly stratified algebra algebra w.r.t. the ordering index set $\underline{e}$.

Proof. Let $\underline{e}_i = (e_{i1}, \cdots, e_{is_i})$ be the complete set of primitive idempotents of A_i and $P_i(j) = A_i e_{ij}$ the projective indecomposable A_i-modules. Then the standard A_i-modules, by definition, are

$$\triangle_i = (\triangle_i(1), \cdots, \triangle_i(s_i)),$$

where

$$\begin{aligned}\triangle_i(j) &= \tfrac{P_i(j)}{U_i(j)},\\ U_i(j) &= A_i(\textstyle\sum_{j<h} e_{ih})A_ie_{ij}\\ &= A_i(\textstyle\sum_{k=j+1}^{s_i} e_{ik})A_ie_{ij}\\ &= \textstyle\sum_{\varphi\in\cup_{k=j+1}^{s_i} Hom(A_ie_{ik},A_ie_{ij})} Im\varphi\end{aligned}$$

(1). By construction, $\Lambda = \bigoplus_{i=1}^n A_i \bigoplus \sum_{t=1}^\infty M^{\otimes^t}$. We can view A_i- module $\Delta_i(j)$ as a Λ-module via the algebra quotient $\Lambda \to A_i$, actually by defining

$$\lambda x = \lambda_i x, \forall\lambda = \lambda_1 + \cdots + \lambda_i + \cdots + \lambda_t + \cdots \in \Lambda,\ x \in \Delta_i(j).$$

We will prove the standard $\Lambda-$module $\Delta(i,j)$ is isomorphic to $\Delta_i(j)$ for any i,j. By definition, we have $\Delta(i,j) = \frac{P(i,j)}{U(i,j)}$, where

$$\begin{aligned}P(i,j) &= \Lambda e_{ij}\\ &= (A_1 \oplus \cdots \oplus A_n \oplus \textstyle\sum_{t=1}^\infty M^{\otimes^t})e_{ij}\\ &= A_ie_{ij} \oplus \textstyle\sum_{i=1}^\infty \bigoplus_{i_0\to i_1\to\cdots\to i_t}\, {}_{i_t}M_{i_{t-1}} \otimes \cdots \otimes\ {}_{i_1}M_{i_0}e_{ij}, (i = i_0)\\ U(i,j) &= \textstyle\sum_{\varphi\in Hom(\Lambda e_{kh},\Lambda e_{ij}),(i,j)<(k,h)} Im\varphi\\ &= \textstyle\sum_{\varphi\in\cup_{k=j+1}^{s_i} Hom(A_ie_{ik},A_ie_{ij})} Im\varphi + \textstyle\sum_{\varphi\in\cup_{i<k} Hom(\Lambda e_k,\Lambda e_{ij})} Im\varphi.\end{aligned}$$

For any $0 \neq m = m_t \otimes \cdots \otimes m_1e_{ij} \in\ {}_{i_t}M_{i_{t-1}} \otimes \cdots \otimes {}_{i_1}M_{i_0}e_{ij} \subseteq M^{\otimes^t}$ where $(i =)i_0 \to i_1 \to \cdots \to i_t$ is the path of length t in Γ, then

$$0 \neq me_{ij} = (m_t \otimes \cdots \otimes m_1)e_{ij} \in\ {}_{i_t}M_{i_{t-1}} \otimes \cdots \otimes {}_{i_1}M_{i_0}e_{ij}.$$

We define

$$\varphi_m : \Lambda e_{i_t} \to \Lambda e_{ij}; ae_{i_t} \mapsto ae_{i_t}me_{ij}, (i < k),$$

then $\varphi_m \in Hom(\Lambda e_k, \Lambda e_{ij})$. Thus

$$\sum_{i=1}^\infty \bigoplus_{i_0\to i_1\to\cdots\to i_t} {}_{i_t}M_{i_{t-1}} \otimes \cdots \otimes\ {}_{i_1}M_{i_0}e_{ij} \subseteq \sum_{\varphi\in\cup_{i<k} Hom(\Lambda e_k,\Lambda e_{ij})} Im\varphi.$$

Hence

$$\Delta(i,j) = \frac{\Lambda e_{ij}}{U(i,j)} \cong \frac{A_ie_{ij}}{U_i(j)} = \Delta_i(j).$$

(2). Now we prove $P(i,j)$ belong to $\mathcal{F}_\Lambda(\Delta)$. By definition

$$\begin{aligned}P(i,j) &= \Lambda e_{ij}\\ &= Aie_{ij} \oplus \sum_{i=1}^\infty \bigoplus_{i_0\to i_1\to\cdots\to i_t} {}_{i_t}M_{i_{t-1}} \otimes \cdots \otimes\ {}_{i_1}M_{i_0}e_{ij}, (i = i_0).\end{aligned}$$

Since A_i is Standardly stratified , we have that $A_i e_{ij} = P_i(j) \in \mathcal{F}_{A_i}(\Delta_i) \subset \mathcal{F}_\Lambda(\Delta)$. We show that ${}_{i_t}M_{i_{t-1}} \otimes \cdots \otimes {}_{i_2}M_{i_1} \otimes {}_{i_1}M_{i_0}$ is a free left $A_{i_t}-$module. Since ${}_{i_1}M_{i_0}$ is free $A_{i_1} - A_{i_0}-$ bimodule, ${}_{i_1}M_{i_0}$ is free left $A_{i_1} \otimes A_{i_0}-$module. We can assume free rank of ${}_{i_1}M_{i_0}$ is l, then

$${}_{i_1}M_{i_0} \cong (A_{i_1} \otimes_k A_{i_0})^l, \text{ for } l \in \mathbf{N}.$$

Then $A_{i_1} \otimes_k A_{i_0} = A_{i_1} \otimes K^h \cong (A_{i_1} \otimes K)^h \cong {A_{i_1}}^h$ where h is the dimension of A_{i_0}. It follows that

$${}_{i_1}M_{i_0} \cong (A_{i_1})^{hl}$$

is a free left $A_{i_1}-$module. Also

$$\begin{aligned} {}_{i_2}M_{i_1} \otimes_{A_{i_2}} {}_{i_1}M_{i_0} &\cong {}_{i_2}M_{i_1} \otimes_{A_{i_1}} (A_{i_1})^{hl} \\ &\cong ({}_{i_2}M_{i_1} \otimes_{A_{i_1}} A_{i_1})^{hl} \cong ({}_{i_2}M_{i_1})^{hl}. \end{aligned}$$

Similarly ${}_{i_2}M_{i_1}$ is free left $A_{i_2}-$module, therefore ${}_{i_2}M_{i_1} \otimes_{A_{i_2}} {}_{i_1}M_{i_0}$ is free left $A_{i_2}-$module. An easy induction shows that ${}_{i_t}M_{i_{t-1}} \otimes \cdots \otimes {}_{i_1}M_{i_0}$ is a free left $A_{i_t}-$module. We may suppose that free rank of ${}_{i_t}M_{i_{t-1}} \otimes \cdots \otimes {}_{i_1}M_{i_0}$ is f. then ${}_{i_t}M_{i_{t-1}} \otimes \cdots \otimes {}_{i_1}M_{i_0}e_{ij}$ is finite rank and free left $A_{i_t}-$module. Hence

$${}_{i_t}M_{i_{t-1}} \otimes \cdots \otimes {}_{i_1}M_{i_0}e_{ij} \cong A_{it}^f \in \mathcal{F}_{A_{i_t}}(\Delta) \subseteq \mathcal{F}_\Lambda(\Delta).$$

Then

$$P(i,j) = A_i e_{ij} \oplus \sum_{t=1}^{\infty} \bigoplus {}_{i_t}M_{i_{t-1}} \otimes \cdots \otimes {}_{i_1}M_{i_0}e_{ij} \in \mathcal{F}_\Lambda(\Delta)$$

for $i = 1, \cdots, n$, $j = 1, \cdots, s_i$.

Combining (1) and (2), we have $\Lambda = K(\Gamma, \mathcal{A})$ is a Standardly stratified. The proof is finished.

Since quasi-hereditary algebras are standardly stratified algebras with endomorphism rings of $\Delta(i)$ being division rings, as a corollary, we have that if all algebras A_i are quasi-hereditary, then the generalized algebra $\Lambda = K(\Gamma, \mathcal{A})$ is also quasi-hereditary. Moreover, we can get a description of $\Delta-$good modules over this quasi-hereditary algebra by the terms of $\Delta-$good modules of these given quasi-hereditary algebras.

Corollary 2. Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a finite connected quiver without oriented cycles. Let $\mathcal{A} = \{A_i | i \in \Gamma_0\}$ be a family of quasi-hereditary algebra (A_i, E_i), $i \in \Gamma_0$. Then $\Lambda = K(\Gamma, \mathcal{A})$ is a quasi-hereditary algebra w.r.t. the ordering index set $\underline{e}$. Moreover

$$\mathcal{F}_\Lambda(\Delta) = \mathcal{F}_{A_1}(\Delta) \int \mathcal{F}_{A_2}(\Delta) \int \cdots \int F_{A_n}(\Delta),$$

which is by definition, the class of $\Lambda-$modules M with a filtration $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_{n-1} \supseteq M_0 = 0$, such that $M_{i-1}/M_i \in \mathcal{F}_{A_i}(\Delta)$.

Proof. In the following, we keep the notations in proof of Theorem 1. By Theorem 1 and its proof, we know that the standard $\Lambda-$module $\Delta(i,j)$ is isomorphic to standard A_i-module $\Delta_i(j)$ for any i,j, and $P(i,j)$ belong to $\mathcal{F}_\Lambda(\Delta)$. It follows that $\mathrm{End}_\Lambda(\Delta(i,j)) \cong \mathrm{End}_{A_i}(\Delta_i(j))$, and then by the quasi-heredity of A_i, $\mathrm{End}_\Lambda(\Delta(i,j))$ is a division ring. Therefore $\Lambda = K(\Gamma, \mathcal{A})$ is a quasi-hereditary algebra w.r.t. the ordering index set $\underline{e}$. Now we prove the second conclusion. From (1) in the proof of Theorem 1, we know that $\Lambda-$module $\Delta(i,j) \simeq \Delta_i(j)$ for any i,j. It follows that

$$\mathcal{F}_{A_1}(\Delta) \int \mathcal{F}_{A_2}(\Delta) \int \cdots \int \mathcal{F}_{A_n}(\Delta) \subseteq \mathcal{F}_\Lambda(\Delta).$$

Now for any $X \in \mathcal{F}_\Lambda(\Delta)$, X admits a $\Delta-$filtration:

$$X = X_{1,0} \supseteq X_{1,1} \cdots \supseteq X_{1,s_1} = X_{2,0} \supseteq X_{2,1} \cdots X_{n,0} \supseteq \cdots X_{n,s_n} = 0,$$

with $X_{i,j-1}/X_{i,j} \simeq \Delta(i,j)^{t_{ij}}$. Then we can get a filtration of $X:\ X = X_{1,0} \supseteq X_{2,0} \cdots X_{n,0} \supseteq \cdots X_{n,s_n} = 0$ with $X(i-1,0)/X(i,0) \in \mathcal{F}_{A_i}(\Delta)$. It follows that $X \in \mathcal{F}_\Lambda(\Delta)$. Then

$$\mathcal{F}_{A_1}(\Delta) \int \mathcal{F}_{A_2}(\Delta) \int \cdots \int F_{A_n}(\Delta) \supseteq \mathcal{F}_\Lambda(\Delta).$$

The proof is finished.

If we take the quiver Γ is $\overrightarrow{A_2}$. we get one of main results in [14].

Corollary 3(Zhu)[14]. Let $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ be the triangular matrix algebra, where ${}_AM$ is free left $A-$module, A and B is quasi-hereditary algebras. Then Λ is quasi-hereditary algebras, $\mathcal{F}_\Lambda(\Delta) = \{(X,Y,f)|X \in \mathcal{F}_A(\Delta),\ Y \in \mathcal{F}_B(\Delta)\}$.

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