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RESEARCH ARTICLE

A construction of dual box

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Abstract. Let $\mathtt{R}$ be a quasi-hereditary algebra, $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ its categories of good and cogood modules correspondingly. In [6] these categories were characterized as the categories of representations of some boxes $\mathcal{A} = \mathcal{A}_\Delta$ and $\mathcal{A}_\nabla$. These last are the box theory counterparts of Ringel duality ([8]). We present an implicit construction of the box $\mathcal{B}$ such that $\mathcal{B} - \text{mo}$ is equivalent to $\mathcal{F}(\nabla)$.

Introduction

Throughout this paper, $\Bbbk$ is an algebraically closed field, all algebras and categories are defined over $\Bbbk$ and the word "module" means "left module". Also we follow the notation from [6].

In the fundamental paper [2] a quasi-hereditary algebra $\mathtt{R}$ has been characterized by two homologically dual subcategories $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ in its module category $\mathtt{R}$ – mod. In [8] was observed, that these categories define an involution (*Ringel duality*) on the classes of Morita equivalence of quasi-hereditary algebras. On other hand, in [6] using the construction of [1] has been developed an alternative approach to the theory of quasi-hereditary algebras. Following [6], a finite dimensional algebra $\mathtt{R}$ is quasi-hereditary if and only if it is Morita equivalent to the Butler-Burt algebra ([1]) of some directed box $\mathcal{A}$. Moreover, in this case the category $\mathcal{F}(\Delta)$ is equivalent to $\mathcal{A} - \text{mod}$ as an exact category. This construction allows to extend many notions and theorems from the case of quasi-hereditary algebras to wider classes of algebras. In particular, in some restrictions on

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the box $\mathcal{A}$, in [6] was constructed a generalization of the Ringel duality. It leads to the notion of a dual box $\mathcal{A}_\nabla$ of a finite dimensional normal box $\mathcal{A} = (A, V)$ with a free kernel $\bar{V}$[1] as a box with the following property: the category $\mathcal{F}(\nabla)$ is equivalent to the category of representation $\mathcal{A}_\nabla - \mathrm{mod}$ as an exact category.

In this paper, starting from the box $\mathcal{A}$, such that $\mathcal{A}-\mathrm{mod}$ is equivalent to the category $\mathcal{F}(\Delta)$, we give an explicit construction of the box $\mathcal{B}$, such that $\mathcal{B} - \mathrm{mod}$ is equivalent to the category $\mathcal{F}(\nabla)$.

The plan of the paper is the following. We assume the box $\mathcal{A}$ is given by its differential graded category (DGC) $\bar{\mathcal{U}} = (A[\bar{V}], \partial)$. In the section 1 we construct DGC $\mathcal{V}$, which defines a completed box $\mathcal{B}$. The rest of the paper is devoted to the construction of an equivalence $\mathcal{B}-\mathrm{mod}$ and $\mathcal{F}(\nabla)$ (Theorem 1). In the section 2 we introduce a category $N(\mathcal{B})$, which turns out to be equivalent to $\mathcal{B} - \mathrm{mod}$ (Lemma 1). In section 3 we construct equivalent to $\mathcal{F}(\nabla)$ subcategory $\mathcal{N}(\mathrm{P}^\bullet)$ in the homotopic category $\mathrm{K}^-(\mathcal{A})$ of complexes over $\mathcal{A}-\mathrm{mod}$. At last (Lemma 5 and Lemma 4) we construct an equivalence of the categories $N(\mathcal{B})$ and $\mathcal{N}(\mathrm{P}^\bullet)$.

1. Main construction

Let $\mathcal{A} = (A, V)$ be a finite dimensional normal box with a free kernel $\bar{V}$, $\mathbb{L} = \mathbb{L}_A$ the category formed by all scalar morphisms in A, $\bar{\mathcal{U}} = A[\bar{V}]$ be the corresponding DGC with the differential $\partial : \bar{\mathcal{U}} \to \bar{\mathcal{U}}$. The canonical embedding $\imath : \mathbb{L} \hookrightarrow A$ induces the following A-bimodule morphisms:

$$m_A : A \otimes_{\mathbb{L}} A \to A; m_l : A \otimes_{\mathbb{L}} \bar{V} \to \bar{V}, m_r : \bar{V} \otimes_{\mathbb{L}} A \to \bar{V}; \tag{1.1}$$

$$m_L : A \otimes_{\mathbb{L}} (\bar{V} \otimes_A \bar{V}) \to \bar{V} \otimes_A \bar{V}, m_R : (\bar{V} \otimes_A \bar{V}) \otimes_{\mathbb{L}} A \to \bar{V}; \tag{1.2}$$

$$m_{\bar{V}} : \bar{V} \otimes_{\mathbb{L}} \bar{V} \to \bar{V} \otimes_A \bar{V}. \tag{1.3}$$

Besides denote the restriction of ∂ on A and $\bar{V}$ by $\partial_0 : A \to \bar{V}$ and $\partial_1 : \bar{V} \to \bar{V} \otimes_A \bar{V}$. For finite dimensional $\mathbb{L}$-bimodules X, Y denote by $p_{X,Y}$ the canonical $\mathbb{L}$-bimodule isomorphism $p_{X,Y} : \mathbb{D}(X \otimes_{\mathbb{L}} Y) \simeq \mathbb{D}(Y) \otimes_{\mathbb{L}} \mathbb{D}(X)$, where $\mathbb{D}$ is the functor of duality over $\Bbbk$. Set

$$N = \{N_i\}_{i\in\mathbb{Z}}, N_1 = \mathbb{D}A, N_0 = \mathbb{D}\bar{V},\ N_{-1} = \mathbb{D}(\bar{V} \otimes_A \bar{V});\ N_i = 0, i \neq 0, \pm 1.$$

Proposition 1. Let $T = \widehat{\mathbb{L}[N]}$. The $\mathbb{L}$-bimodule morphisms

$$d_T|_{N_1} = p_{AA}\mathbb{D}m_A,\ d_T|_{N_0} = -p_{A\,\bar{V}}\mathbb{D}m_l + p_{\bar{V}\,A}\mathbb{D}m_r + \mathbb{D}\partial_0, \tag{1.4}$$

$$d_T|_{N_{-1}} = p_{A\ \bar{V}\otimes\bar{V}}\mathbb{D}m_L + p_{\bar{V}\otimes\bar{V}\ A}\mathbb{D}m_R + p_{\bar{V}\,\bar{V}}\mathbb{D}m_{\bar{V}} + \mathbb{D}\partial_1 \tag{1.5}$$

defines on T the structure of completed DGC.

[1]The proof of the uniqueness of $\mathcal{A}_\nabla$ will be published elsewhere.

Proof. The Leibniz rule and continuity allows to extend d to the $\mathbb{L}$-bimodule map $d : T \to T$. It leaves to prove $d^2(N) = 0$.

The structure of DGC on $\bar{\mathfrak{U}}$ gives the DGC structure on $\check{\mathfrak{U}}$, $\check{\mathfrak{U}} = \bar{\mathfrak{U}}/\sum_{i\geq 3}\bar{\mathfrak{U}}_i$. We will identify $\check{\mathfrak{U}}$ with the sum $\bar{\mathfrak{U}}_0 \oplus \bar{\mathfrak{U}}_1 \oplus \bar{\mathfrak{U}}_2$ of the components of degree 1 and 2 of $\bar{\mathfrak{U}}$. In turn, the DGC structure on $\check{\mathfrak{U}}$ defines the structure of an $A(\infty)$-category over $\mathbb{L}$ on $\check{\mathfrak{U}}$ ([4]). More precisely, $\check{\mathfrak{U}}$ is endowed with a family of multiplications $(m_1, m_2, \dots)$, $m_i : M^{\otimes_{\mathbb{L}} i} \to M$ of degree $+1$, $m_1 = d (= d_{\check{\mathfrak{U}}})$, $m_2(u_1 \otimes u_2) = (-1)^{\deg u_1} u_1 u_2$, $m_i = 0$ for $i \geq 3$. The multiplications $m_i, i \geq 1$ should satisfy certain axioms. These axioms can be united by so called bar-construction, which endows the tensor cocategory $\mathcal{T}^+ = \bigoplus_{i=1}^{\infty} s(\check{\mathfrak{U}})^{\otimes_{\mathbb{L}} i}$ with a $\mathbb{L}$-linear codifferential $\delta : \mathcal{T}^+ \to \mathcal{T}^+$, where s is the grading shift (see [4] for details). Then applying the functor of $\Bbbk$-duality $\mathbb{D}$ we obtain on the completed precategory (i.e. category without units) $T^+ = \prod_{i=1}^{\infty} \mathbb{D}(s(\check{\mathfrak{U}}))^{\otimes_{\mathbb{L}} i}$ the differential $\mathbb{D}(d) : T^+ \to T^+$, coinciding with the differential d_T, given by (1.4) and (1.5). Then the condition $d_T^2 = 0$ is just the dual to the condition $d^2 = 0$. □

Following [6], [7] T defines the positively graded DGC $\mathcal{V} = T/I$, where I is the differential ideal, generated by N_{-1}. As a category $\mathcal{V}$ is freely generated over $B = T_0/(T_0 \cap I)$ by N_1. The corresponding completed box $\mathcal{B} = (B, W)$ is by construction normal and weakly triangular.

The main theorem of this paper is the following.

Theorem 1. $\mathcal{B} - \mathrm{mod}$ is equivalent to $\mathcal{F}(\nabla)$.

We do not prove here the uniqueness of $\mathcal{A}_\nabla$, since the proof uses techniques of $A(\infty)$-categories. This fact is closely related with the question of uniqueness of a minimal exact Borel subalgebra in a class of Morita equivalence of quasi-hereditary algebras (see [5], [6]). Another issue is the generalization of Ringel duality, which needs finite dimensionality of $\mathcal{A}_\nabla$. The last condition often can be checked using the presented construction of $\mathcal{B}$. In particular, if $\mathcal{A}$ is directed, then $\mathcal{B}$ is directed as well.

2. A realization of representations category

Every $M \in \mathcal{B} - \mathrm{mod}$ is an object of $\mathbb{L}[\mathbb{D}\bar{V}] - \mathrm{mod}$, hence it can be considered as a left $\mathbb{L}$-module $M = \{M(\mathtt{i}) | \mathtt{i} \in \mathrm{Ob}\, A\}$. The structure of a $\mathbb{L}[\mathbb{D}\bar{V}]$-module on $\mathbb{L}$-module M is given by a $\mathbb{L}$-bimodule map s_M :

$\mathbb{D}\bar{V} \to \mathrm{Hom}_{\Bbbk}(M,M)$. Since

$$\mathrm{Hom}_{\mathbb{L}-\mathbb{L}}(\mathbb{D}\bar{V}, \mathrm{Hom}_{\Bbbk}(M,M)) \simeq \mathrm{Hom}_{\mathbb{L}}(M, \bar{V} \otimes_{\mathbb{L}} M), \qquad (2.6)$$

s_M is uniquely defined by a $\mathbb{L}$-module homomorphism $c_M : M \to \bar{V} \otimes_{\mathbb{L}} M$.

The $\mathbb{L}[\mathbb{D}\bar{V}]$-module M is a B-module only if it vanishes on the relations, defined by $d_{\mathbb{L}[N]}|_{N_{-1}}$, i.e. by (1.5),

$$s_M \mathbb{D}\partial + m_{\mathbb{L}}(s_M \otimes s_M) p_{\bar{V}\,\mathbb{D}\bar{V}} \mathbb{D} m_{\bar{V}} = 0.$$

Using the isomorphism (2.6), we can rewrite this condition as

$$(\partial \otimes \mathbb{1}_M) c_M + (m_{\bar{V}} \otimes \mathbb{1}_M)(\mathbb{1}_{\bar{V}} \otimes c_M) c_M = 0. \qquad (2.7)$$

In this assumption M possesses a structure of B-module if and only if in M exists a full flag (a composition series over $\mathbb{L}$) $\{M_i \,|\, i = 0, \dots, n = n(M)\}$ in M, such that $s_M(\mathbb{D}\bar{V})(M_i) \subset M_{i-1}$, equivalently

$$c_M(M_i) \subset \bar{V} \otimes M_{i-1}, i = 1, \dots, n. \qquad (2.8)$$

Let $M, N \in \mathcal{B} - \mathrm{mod}$. Then any morphism $f : M \to N$ is defined by $s_f \in \mathrm{Hom}_{\mathbb{L}-\mathbb{L}}(\mathbb{D}A, \mathrm{Hom}_{\Bbbk}(M,N))$, which, following the definition (1.4) of $d_T|_{N_0}$, should satisfy the relation ([7], ?)

$$m\big((s_f \otimes s_M)(p_{\bar{V}\,A})\mathbb{D}m_r - (s_N \otimes s_f)(p_{A\,\bar{V}})\mathbb{D}m_l\big) + s_M \mathbb{D}\partial = 0, \qquad (2.9)$$

where m is the morpisms composition in the category of $\mathbb{L}$-modules.

As above, by the canonical isomorphism

$$\mathrm{Hom}_{\mathbb{L}-\mathbb{L}}(\mathbb{D}A, \mathrm{Hom}_{\Bbbk}(M,N)) \simeq \mathrm{Hom}_{\mathbb{L}}(M, A \otimes_{\mathbb{L}} N) \qquad (2.10)$$

s_f corresponds to the $\mathbb{L}$-module morphism $c_f : M \to A \otimes_{\mathbb{L}} N$ and the condition (2.9) can be rewritten as

$$\big(-(m_l \otimes \mathbb{1}_N)(\mathbb{1}_A \otimes c_N) + (\partial \otimes \mathbb{1}_N)\big) c_f + (m_r \otimes \mathbb{1}_N)(\mathbb{1}_{\bar{V}} \otimes c_f) c_M = 0. \quad (2.11)$$

Assume morphisms $f : M \to N$ and $g : N \to S$ are given by corresponding s_f, s_g as above. Then by the definition (1.4) of $d_T|_{N_1}$ the $\mathbb{L}$-bimodule morphism s_{gf}, corresponding to the composition $gf : M \to S$ is just the composition

$$s_{gf} = m(s_g \otimes s_f) p_{AA} \mathbb{D} m_A. \qquad (2.12)$$

If the morphism f, g from $\mathcal{B}-\mathrm{mod}$ are presented as $c_f \in \mathrm{Hom}_{\mathbb{L}}(M, A \otimes_{\mathbb{L}} N)$ and $c_g \in \mathrm{Hom}_{\mathbb{L}}(N, A \otimes_{\mathbb{L}} S)$, then the equality (2.12) can be rewritten as

$$c_{g\,f} = (m_A \otimes \mathbb{1}_S)(\mathbb{1}_A \otimes c_g) c_f. \qquad (2.13)$$

Let $N(\mathcal{B})$ be a category, which objects are the triples $(M, \{M_i\}, c_M)$, where $M \in \mathbb{L} - \text{mod}$, $\{M_i\}$ is a full flag in M and a morphism c_M, satisfies (2.7), (2.8). The morphisms in $N(\mathcal{B})$ are defined as above by c_f satisfying the condition (2.11) and the composition of morphisms is defined by (2.13).

Lemma 1. The categories $\mathcal{B} - \text{mod}$ and $N(\mathcal{B})$ are equivalent.

Proof. Define the functor $c : \mathcal{B} - \text{mod} \to N(\mathcal{B})$ as follows. If $M \in \operatorname{Ob} \mathcal{B} - \text{mod}$, then for $R = \operatorname{Rad} B$ gives us the following strictly descent chain of L-submodules

$$M \supset RM \supset R^2M \supset \cdots \supset R^nM = 0 \tag{2.14}$$

for some $n \geq 1$. Then we set $c(M) = (M|_{\mathbb{L}}, c_M, \{M_i\})$, where c_M is defined above and $\{M_i\}$ is a refinement of the chain (2.14). Note, that the isoclass of $c(M)$ in $N(\mathcal{B})$ does not depend on the choice of refinement. If $f : M \to N$ is a morphism from $\mathcal{B} - \text{mod}$, then we set $c(f) = c_f$. The isomorphisms (2.6) and (2.10) above show that c is a full and faithful functor. Using the same isomorphisms (2.6) and (2.10) one can define the quasi-inverse to c functor $s : N(\mathcal{B}) \to \mathcal{B} - \text{mod}$. □

3. Category of cogood modules

Sometimes we will abuse notations and will skip $i \in \mathbb{Z}$ in the notation like ∂^i_M in the differential of the complex $M^\bullet$ etc.

Let $\operatorname{Ob} A = \{\mathtt{1}, \dots, \mathtt{n}\}$ be the set of objects of A. Recall, that the category $\mathcal{F}(\nabla)$ is an extension closure of the set of costandard modules $\{\nabla_\mathtt{1}, \dots, \nabla_\mathtt{n}\}$, ([2]). We construct some categories of complexes over $\mathcal{A}$ equivalent to $\mathcal{F}(\nabla)$. Let $\mathtt{R}$ be the right Butler-Burt algebra of $\mathcal{A}$, $F : \mathcal{A} - \text{mod} \to \mathtt{R} - \text{mod}$ the Burt-Butler functor and $D(F) : D(\mathcal{A}) \to D(\mathtt{R})$ the induced derived functor ([6]). For any $\mathtt{i} \in \operatorname{Ob} A$ in [6] is constructed a K_Ω-injective complex $\mathrm{I}^\bullet_\mathtt{i} \in D^-(\mathcal{A})$, such that $D(F)(\mathrm{I}^\bullet_\mathtt{i}) \simeq \nabla_\mathtt{i}$, in particular $D(F)$ induced an equivalence between the triangular subcategories in $D(\mathcal{A})$ and $D(\mathtt{R})$, generated by all $\mathrm{I}^\bullet_\mathtt{i}$ and $\nabla_\mathtt{i}$ correspondingly, $\mathtt{i} \in \operatorname{Ob} A$.

For us will be more convenient instead of the subcategory in $D(\mathcal{A})$, generated by $\mathrm{I}^\bullet_\mathtt{i}$ consider the isomorphic subcategory, generated by $\mathrm{P}^\bullet_\mathtt{i}$, $\mathtt{i} \in \operatorname{Ob} A$ ([6], Section 2). Denote $\mathrm{P}^\bullet = \oplus_{\mathtt{i} \in \operatorname{Ob} A} \mathrm{P}^\bullet_\mathtt{i}$. Recall, that $\mathrm{P}^\bullet$ is a positive complex and $\mathrm{P}^i = \bar{V}^i$, $i \geq 0$ ($\bar{V}^0 = A$) and $\partial_\mathrm{P}(\omega_\mathtt{i})(x) = -\partial(x)$, $\partial_\mathrm{P}(\varphi)(x) = \varphi x$, provided the right side is defined.

Let $\mathcal{C}(\mathrm{P}^\bullet)$ be a minimal full extension closed subcategory in $D(\mathcal{A})$ containing $\mathrm{P}^\bullet_\mathtt{i} \in \mathcal{C}(\mathrm{P}^\bullet)$ for any $\mathtt{i} \in \operatorname{Ob} A$, i.e. for any triangle

$$X^\bullet \xrightarrow{i} Y^\bullet \xrightarrow{p} Z^\bullet \to X^\bullet[1] \tag{3.15}$$

from $X^\bullet, Z^\bullet \in \mathcal{C}(\mathrm{P}^\bullet)$ follows $Y^\bullet \in \mathcal{C}(\mathrm{P}^\bullet)$. By construction the categories $\mathcal{F}(\nabla)$ and $\mathcal{C}(\mathrm{P}^\bullet)$ are equivalent. Since $\mathrm{P}^\bullet_{\mathtt{i}}$ are K_Ω-projective, the category $\mathcal{C}(\mathrm{P}^\bullet)$ consists of K_Ω-projective complexes, that allows us to calculate in this category the morphisms in $\mathrm{K}(\mathcal{A})$ instead of $D(\mathcal{A})$.

Next we consider the category $\mathcal{N}'(\mathrm{P}^\bullet)$, which objects are $M^\bullet \in \mathcal{C}(\mathrm{P}^\bullet)$ endowed with a filtration of the objects from $\mathcal{N}'(\mathrm{P}^\bullet)$

$$0 = M^\bullet_0 \subset M^\bullet_1 \subset \cdots \subset M^\bullet_{n-1} \subset M^\bullet_n = M^\bullet, \qquad (3.16)$$

such that $M^\bullet_i \simeq \mathrm{Cone}(e_i)$ for some $e_i : \mathrm{P}^\bullet_{\mathtt{i}_i}[-1] \to M^\bullet_{i-1}, i = 1, \ldots, n, \mathtt{i}_i \in \mathrm{Ob}\, A$ (we assume zero complex also belongs to $\mathcal{N}'(\mathrm{P}^\bullet)$. The morphisms in $\mathcal{N}'(\mathrm{P}^\bullet)$ does not depend on the filtration and are the same as in $\mathcal{C}(\mathrm{P}^\bullet)$. The number $n = l(M^\bullet)$ we call the length of $M^\bullet$. Due to normality $\mathcal{A}$ this number is correctly defined.

Lemma 2. If $N^\bullet_1 \xrightarrow{f_1} N^\bullet_2 \xrightarrow{f_2} N^\bullet_3$ is a sequence in $\mathrm{Com}(\mathcal{A})$, h is a homotopy between $f_2 f_1$ and 0, then it defines the morphisms

$$g_1 = g_1(f_1, f_2, h) : \mathrm{Cone}(f_1) \to N^\bullet_3, g^i_1 = \begin{pmatrix} h^{i+1} & f^i_2 \end{pmatrix}; \qquad (3.17)$$

$$g_2 = g_2(f_1, f_2, h) : N^\bullet_1[1] \to \mathrm{Cone}(f_2), g^i_2 = \begin{pmatrix} -f^{i+1}_1 \\ h^{i+1} \end{pmatrix} \qquad (3.18)$$

such that

$$f_1[1] : N^\bullet_1[1] \xrightarrow{g_2[1]} \mathrm{Cone}(f_2) \xrightarrow{p} N_2[1], \qquad (3.19)$$

$$f_2 : N^\bullet_2 \xrightarrow{i} \mathrm{Cone}(f_1) \xrightarrow{g_1} N^\bullet_3, \qquad (3.20)$$

where i and p are the canonical homomorphism.

In opposite, if g_1 (g_2) satisfies (3.19) ((3.20)), then g_1 (g_2) has a form (3.17) ((3.18)). If $\mathrm{K}(\mathcal{A})(N^\bullet_1[1], N^\bullet_3) = 0$, then g_1 and g_2 are defined uniquely up to homotopy. Besides, there exists a canonical isomorphisms $\Phi : \mathrm{Cone}(g_1) \simeq \mathrm{Cone}(g_2)$.

Proof. Immediately is checked, that g_1 and g_2 are homomorphisms of complexes, satisfying (3.19) and (3.20) and the opposite statement.

In the complexes $\mathrm{Cone}(g_1)$ and $\mathrm{Cone}(g_2[-1])$ the i-th component equals $N^{i+2}_1 \oplus N^{i+1}_2 \oplus N^i_3$ and i-th differential has a matrix

$$\begin{bmatrix} \partial^{i+2}_{N_1} & 0 & 0 \\ -f^{i+2}_1 & -\partial^{i+1}_{N_2} & 0 \\ h^{i+2} & f^{i+1}_2 & \partial^i_{N_3} \end{bmatrix},$$

that gives us the isomorphism Ψ.

We prove the uniqueness statement for g_1, the case of g_2 is treated analogously. Consider the triangle

$$\cdots \to N_1^\bullet \xrightarrow{f_1} N_2^\bullet \xrightarrow{i} \mathrm{Cone}(f_1) \longrightarrow N_1^\bullet[1] \to \dots .$$

Applying $\mathrm{K}(\mathcal{A})(\_, N_3^\bullet)$ we obtain the exact sequence

$$0 = \mathrm{K}(\mathcal{A})(N_1^\bullet[1], N_3^\bullet) \to \mathrm{K}(\mathcal{A})(\mathrm{Cone}(f_1), N_3^\bullet) \to$$
$$\to \mathrm{K}(\mathcal{A})(N_2^\bullet, N_3^\bullet) \to \mathrm{K}(\mathcal{A})(N_1^\bullet, N_3^\bullet).$$

Since the second arrow is mono, it gives us the uniqueness of g_1. □

Proposition 2. The category $\mathcal{N}'(\mathrm{P}^\bullet)$ is equivalent to $\mathcal{C}(\mathrm{P}^\bullet)$.

Proof. To prove the equivalence there is enough to check, that every object $M^\bullet$ from $\mathcal{C}(\mathrm{P}^\bullet)$ is isomorphic to an object $N^\bullet$ from $\mathcal{N}'(\mathrm{P}^\bullet)$. We prove it by induction on $l(M^\bullet)$. The base $l(M^\bullet) = 1$ is obvious.

For the induction step from n to $n+1$ assume $M^\bullet = \mathrm{Cone}(K^\bullet[-1] \xrightarrow{f} L^\bullet)$, $K^\bullet, L^\bullet$ are nonzero complexes in $\mathcal{N}'(\mathrm{P}^\bullet)$, $l(M^\bullet) = n+1$. By induction we can assume $K^\bullet = \mathrm{Cone}(f_1)$ for some $f_1 : \mathrm{P}_{\mathtt{i}}^\bullet[-1] \to N^\bullet$. Applying $\mathrm{K}_{\mathcal{A}}(\_, L^\bullet)$ to the exact triangle

$$\cdots \to \mathrm{P}_{\mathtt{i}}^\bullet[-2] \xrightarrow{f_1} N^\bullet[-1] \xrightarrow{f_2} K^\bullet[-1] \to \mathrm{P}_{\mathtt{i}}^\bullet[-1] \to \dots$$

we obtain the sequence

$$\mathrm{K}(\mathcal{A})(K^\bullet[-1], L^\bullet) \xrightarrow{\pi} \mathrm{K}(\mathcal{A})(N^\bullet[-1], L^\bullet) \xrightarrow{\sigma} \mathrm{K}(\mathcal{A})(\mathrm{P}_{\mathtt{i}}^\bullet[-2], L^\bullet). \quad (3.21)$$

Since $\sigma\pi(f) = 0$ it gives us the sequence

$$\mathrm{P}_{\mathtt{i}}^\bullet[-2] \xrightarrow{f_1[-1]} N^\bullet[-1] \xrightarrow{f_2} L^\bullet$$

and the homotopy h between $f_2 f_1$ and 0, such that $g_1 = g_1(f_1, f_2, h) = f$. By Lemma 2 holds $M^\bullet \simeq \mathrm{Cone}(g_2)$, $g_2 = g_2(f_1, f_2, h)$, $g_2 : \mathrm{P}_{\mathtt{i}}^\bullet[-1] \to \mathrm{Cone}(f_2)$. By induction $\mathrm{Cone}(f_2)$ is isomorphic to some $M_1^\bullet \in \mathcal{N}'(\mathrm{P}^\bullet)$, hence $M^\bullet \simeq \mathrm{Cone}(\mathrm{P}_{\mathtt{i}}^\bullet \to M_1^\bullet)$ belongs to $\mathcal{N}'(\mathrm{P}^\bullet)$. □

For $M^\bullet \in \mathcal{N}'(\mathrm{P}^\bullet)$ define inductively a $\mathbb{L}$-submodule M in M^0 as follows: if $M^\bullet = \mathrm{P}_{\mathtt{i}}^\bullet$, then we set $M = \Bbbk \cdot \mathbb{1}_{\mathtt{i}}$ and if $M = \mathrm{Cone}(e)$ for $e \in \mathrm{Com}(\mathcal{A})(\mathrm{P}_{\mathtt{i}}^\bullet, N^\bullet)$, $\mathtt{i} \in \mathrm{Ob}\, A$, $N^\bullet \in \mathcal{N}(\mathrm{P}^\bullet)$, then set $M = \Bbbk \cdot \mathbb{1}_{\mathtt{i}} \oplus N$. By the construction M is endowed with the canonical full $\mathbb{L}$-flag $\{M_i\}, i = 0, \dots, \dim_\Bbbk M$. Note, that there exists the canonical isomorphism of graded $\mathbb{L}$-bimodules $M^\bullet \simeq \mathrm{P}^\bullet \otimes_{\mathbb{L}} \mathrm{top}(M^\bullet)$.

Denote for $M^\bullet, N^\bullet$ by $\mathrm{Com}_A(\mathcal{A})(M^\bullet, N^\bullet)$ the space of morphisms $f : M^\bullet \to N^\bullet$, such that $f^i : M^i \to N^i, i \in \mathbb{Z}$ belongs to $A - \mathrm{mod}$. Such morphisms form a subcategory $\mathrm{Com}_A(\mathcal{A})$ in $\mathrm{Com}(\mathcal{A})$.

Lemma 3. Let $M^\bullet, N^\bullet \in \mathcal{N}'(\mathrm{P}^\bullet)$, $f \in \mathrm{K}(M^\bullet, N^\bullet)$. Then there exists a unique $\mathrm{f} \in \mathrm{Com}_A(\mathcal{A})(M^\bullet, N^\bullet)$, such that f is homotopic to f. In particular, the subcategory in $\mathcal{N}'(\mathrm{P}^\bullet)$ of $M^\bullet$, such that in the definition of $\mathcal{N}'(\mathrm{P}^\bullet)$ all $e_i \in \mathrm{Com}_A(\mathcal{A})$, is equivalent to $\mathcal{F}(\nabla)$. Besides, f is uniquely defined by f^0.

Proof. We prove the statement by induction on the length. The base of induction is $M^\bullet = \mathrm{P}^\bullet_{\mathtt{i}}$. Following Theorem 1, [6], the homotopy class of $f : \mathrm{P}^\bullet_{\mathtt{i}} \to N^\bullet$ is uniquely defined by $n_f \in \mathrm{Ker}\, \partial^0_N$ and the condition $\mathrm{f}(\bar{V}) = 0$ for all i defines the unique representative f of f in $\mathrm{Com}_A(\mathcal{A})(\mathrm{P}^\bullet_{\mathtt{i}}, N^\bullet)$.

Let $\mathrm{e} \in \mathrm{Com}_A(\mathcal{A})(\mathrm{P}^\bullet_{\mathtt{i}}[-1], L^\bullet)$ be a morphism, such that $M^\bullet = \mathrm{Cone}(\mathrm{e})$, $L^\bullet \in \mathcal{N}'(\mathrm{P}^\bullet)$. The long exact sequence in $\mathrm{K}(\mathcal{A})$ obtained by applying $D(\mathcal{A})(\_, N^\bullet)$ to the corresponding triangle gives

$$\cdots \to 0 \to \mathrm{K}(\mathcal{A})(\mathrm{P}^\bullet_{\mathtt{i}}, N^\bullet) \xrightarrow{\pi} \mathrm{K}(\mathcal{A})(\mathrm{Cone}(\mathrm{e}), N^\bullet) \xrightarrow{\sigma}$$
$$\mathrm{K}(\mathcal{A})(L^\bullet, N^\bullet) \xrightarrow{\delta} \mathrm{K}(\mathcal{A})(\mathrm{P}^\bullet_{\mathtt{i}}[-1], N^\bullet) \to \dots.$$

The morphism δ maps any $\mathrm{g} \in \mathrm{Com}_A(\mathcal{A})(L^\bullet, N^\bullet)$ in $\mathrm{ge} : \mathrm{P}^\bullet_{\mathtt{i}}[-1] \to N^\bullet$. The class of g belongs to $\mathrm{Im}\,\sigma$ if and only if ge is contractible. Recall a description of the layer $\sigma^{-1}(\mathrm{g})$. If $t \in \sigma^{-1}(\mathrm{g})$, then we can construct t by Lemma 2 using the contracting homotopy $\mathrm{h} = \mathrm{h}(t)$. Assume $t' \in \mathrm{K}(\mathcal{A})(\mathrm{Cone}(\mathrm{e}), N^\bullet)$. Then $t' \in \sigma^{-1}(\mathrm{g})$ if and only if $\sigma(t') = \mathrm{g}$ and $\{(t^i - t'^i) \mid i \in \mathbb{Z}\}$ is a homomorphism $\mathrm{P}^\bullet_{\mathtt{i}} \to N^\bullet$. Since any homomorphism of complexes $f : \mathrm{P}^\bullet_{\mathtt{i}} \to N^\bullet$ is defined by $f^0(\omega_{\mathtt{i}})(\mathbb{1}_{\mathtt{i}})$ and $f(\bar{V})$, changing t to t' we can assume $\mathrm{h}(\bar{V}) = 0$. By induction $\mathrm{Com}_A(\mathcal{A})(\mathrm{P}^\bullet_{\mathtt{i}}, N^\bullet)$ is a set of representatives of all homotopy classes from $\mathrm{K}(\mathcal{A})(\mathrm{P}^\bullet_{\mathtt{i}}, N^\bullet)$. Then adding $\mathrm{K}(\mathcal{A})(\mathrm{P}^\bullet_{\mathtt{i}}, N^\bullet)$ to h we obtain all representatives of the homotopy class $\sigma^{-1}(\mathrm{g})$. Besides, since π is a monomorphism, t is homotopic to t', if and only if $t = t'$, hence all classes are non-homotopic. By induction assume, that e and g belongs to $\mathrm{Com}_A(\mathcal{A})$. Then by Lemma 2

$$\mathrm{f} : \mathrm{Cone}(\mathrm{e}) \to N^\bullet,\ \mathrm{f}^i = \begin{pmatrix} \mathrm{h}^{i+1} & \mathrm{g}^i \end{pmatrix}$$

will belong to $\mathrm{Com}_A(\mathcal{A})$. If $\mathrm{f}^0 = 0$, then by induction $\mathrm{g} = 0$. Then $\{\mathrm{h}^{i+1}\}_{i \in \mathbb{Z}}$ is a homomorphism $\mathrm{P}^\bullet \to N^\bullet$, such that $\mathrm{h}^0 = 0$, hence $\mathrm{h} = 0$ and $\mathrm{f} = 0$. □

Denote by $\mathcal{N}(\mathrm{P}^\bullet)$ the subcategory in $\mathcal{N}'(\mathrm{P}^\bullet)$, which objects for the definition (3.16) all e_i-th belongs to $\mathrm{Com}_A(\mathcal{A})$ and for $M^\bullet, N^\bullet \in \mathrm{Ob}\,\mathcal{N}(\mathrm{P}^\bullet)$

$$\mathcal{N}(\mathrm{P}^\bullet)(M^\bullet, N^\bullet) = \mathrm{Com}_A(\mathcal{A})(M^\bullet, N^\bullet) \cap \mathcal{N}'(\mathrm{P}^\bullet)(M^\bullet, N^\bullet). \tag{3.22}$$

By Lemma 3 the category $\mathcal{N}(\mathrm{P}^\bullet)$ is equivalent to $\mathcal{N}'(\mathrm{P}^\bullet)$.

Lemma 4. Any object $M = (M, c_M, \{M_i\})$ of $N(\mathcal{A})$ defines the complex $\mathsf{n}(M) = M^\bullet \in \mathcal{N}(\mathrm{P}^\bullet)$ as follows ($\otimes = \otimes_{\mathbb{L}}$)

$$M^0 \simeq A \otimes M,\ M^i \simeq \underbrace{\bar{V} \otimes_A \cdots \otimes_A \bar{V}}_{i} \otimes M,\ i \geq 1; \tag{3.23}$$

$$\partial^i_M(\omega_{\mathtt{j}})(x \otimes m) = -\partial(x) \otimes m + \hat{x} \otimes_A c_M(m), \hat{x} = (-1)^i x, \tag{3.24}$$
$$x \in \bar{V}^{\otimes i}(\mathtt{i}, \mathtt{j}),\ m \in M,\ \text{where } \partial \text{ is the differential in } \bar{\mathcal{U}},$$
$$\partial^i_M(v)(x \otimes m) = v \otimes_A x \otimes m, v \in \bar{V}(\mathtt{j}, \mathtt{k}),\ \mathtt{i}, \mathtt{j}, \mathtt{k} \in \operatorname{Ob} A.$$

If $M, N \in N(\mathcal{A})$, then any morphism $f \in N(\mathcal{A})(M, N)$ defines unique morphism $\mathrm{f} = \mathsf{n}(f) : \mathsf{n}(M) \to \mathsf{n}(M)$, such that $\mathrm{f}^0|_M = f$, which turns n into a functor $\mathsf{n} : N(\mathcal{A}) \to \mathcal{N}(\mathrm{P}^\bullet)$.

Proof. We prove that the defined above ∂_M's are morphisms from $\mathcal{A}-$mod, i.e. for any $a \in A$ holds $r = \partial_M(\omega_{\mathtt{j}} a - a\omega_{\mathtt{i}} + \partial(a)) = 0$, [7].

$$r(x \otimes m) = -\partial(ax) \otimes m + ax \otimes_A c_M(x) + a\partial(x) \otimes m$$
$$- ax \otimes_A c_M(m) + \partial(a) \otimes x \otimes m = 0$$

by the Leibniz rule. Prove that $M^\bullet$ is a complex, i.e. $\partial^2_M = 0$.

$$\partial^2_M(\omega_{\mathtt{i}})(x \otimes m) = \partial_M(\omega_{\mathtt{i}})\partial_M(\omega_{\mathtt{i}})(x \otimes m) = \partial_M(\omega_{\mathtt{i}})(\partial(x) \otimes m +$$
$$\hat{x} \otimes_A c_M(m)) = \partial^2(x) \otimes m - \widehat{\partial(x)} \otimes_A c_M(m) - \partial(\hat{x}) \otimes_A c_M(m) -$$
$$x \otimes_A (\partial \otimes \mathbb{1}_M)c_M(m) - x \otimes_A m_V(\mathbb{1}_V \otimes c_M)c_M(m) = 0 \text{ due to (2.7).}$$
$$\partial^2_M(\varphi)(x \otimes m) = \partial_M(\omega_{\mathtt{i}})\partial_M(\varphi)(x \otimes m) + \partial_M(\varphi)\partial_M(\omega_{\mathtt{j}})(x \otimes m) +$$
$$\partial_M(\partial(\varphi))(x \otimes m) = \partial(\varphi x) \otimes m + \widehat{\varphi \otimes_A x} \otimes_A c_M(m) -$$
$$\varphi \otimes_A \partial(x) + \varphi \otimes_A \hat{x} \otimes_A c_M(m) + \partial(x) \otimes_A x \otimes_A m = 0$$

due to Leibniz rule.

The filtration of $M^\bullet$ is defined by $M^\bullet_i$ and the e_i-th from definition (3.16) are defined by the second summand in the definition of ∂_M.

To prove the statement about morphisms define $\mathrm{f} = \mathsf{n}(f)$ as $\mathrm{f}(x \otimes m) = x \otimes_A c_f(m)$. We prove, that $\mathsf{n}(f)$ is a morphism of complexes.

$$(\mathrm{f}\partial_M - \partial_M \mathrm{f})(x \otimes m) = \mathrm{f}(-d(x) \otimes m + \hat{x} \otimes_A c_M(m)) -$$
$$\partial_M(x \otimes_A c_f(m)) = \big(-d(x) \otimes_A c_f(m) + \hat{x} \otimes_A (\mathbb{1}_A \otimes c_f)c_M\big) -$$
$$\big(-d(x) \otimes_A c_f(m) - \hat{x} \otimes_A (\partial \otimes \mathbb{1}_M)c_M - \hat{x} \otimes_A (\mathbb{1}_A \otimes c_N)c_f(m)\big) =$$
$$\hat{x} \otimes_A ((\mathbb{1}_A \otimes c_f)c_M - (\partial \otimes \mathbb{1}_M)c_M - (\mathbb{1}_A \otimes c_N)c_f)(m) = 0$$

due to (2.11). □

Obviously, the image of n is a dense subcategory in $\mathcal{N}(\mathrm{P}^\bullet)$.

Lemma 5. Let $M^\bullet, N^\bullet \in \mathcal{N}(\mathrm{P}^\bullet)$, $\mathrm{f} \in \mathcal{N}(\mathrm{P}^\bullet)(M^\bullet, N^\bullet)$. Set

$$c(M^\bullet) = (M, \{M_i\}, c_M), M_i = \mathrm{top}(M_i^\bullet), c_M|_{\mathrm{top}(\mathrm{P}_{\mathtt{i}_i})} = f_i,\ c(\mathrm{f}) = \mathrm{f}^0|_M,$$

where M is considered as a $\mathbb{L}$-submodule in M^0 by $M \simeq \mathbb{L} \otimes_{\mathbb{L}} M \subset A \otimes_{\mathbb{L}} M \simeq M^0$. Then it gives us the functor $c : \mathcal{N}(\mathrm{P}) \to N(\mathcal{B})$.

Proof. c_M satisfies the condition (2.7) follows from $f_n^0(\omega_{\mathtt{i}})(\mathbb{1}_{\mathtt{i}_n})\partial^0_{M_{n-1}} = 0$ is equivalent to (2.7). The formula for the composition (2.13) follows from the formula of composition of morphisms of complexes. □

Lemmas 4 and 5 gives us the following corollary and Theorem 1.

Corollary 1. $\mathsf{n} : N(\mathcal{B}) \to \mathcal{N}(\mathrm{P}^\bullet)$ and $\mathsf{n} : \mathcal{N}(\mathrm{P}^\bullet) \to N(\mathcal{B})$ is mutual quasi-inverse equivalences.

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