RESEARCH ARTICLE

Linear cellular automata: Garden of Eden Theorem, L-surjunctivity and group rings

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Communicated by V. V. Kirichenko

Dedicated to the memory of V. M. Usenko

ABSTRACT.   This paper is a slightly extended version of the lecture given by the first author at the "5th International Algebraic Conference in Ukraine" held on July 20–27 2005 at the Odessa I.I. Mechnikov National University.

1.   Cellular automata

1.1.   Finitely generated groups

Let G be a finitely generated group. Fix a finite generating system S which is symmetric (i.e. $s \in S \Rightarrow s^{-1} \in S$). Then the *Cayley graph* of G w.r. to S is the graph $Cay(G, S)$ with vertex set G and edges $\{g, gs\}$, $g \in G$ and $s \in S$. The *S-length* $\ell_S(g)$ of a group element $g \in G$ is defined by $\ell_S(1_G) = 0$ (where 1_G denotes the unit element in G) and, for $g \neq 1_G$,

$$\ell_S(g) = \min\{n \in \mathbb{N} : g = s_1 s_2 \cdots s_n, \ s_i \in S\}.$$

The finitely generated group G can then be endowed with a metric space structure by introducing the *distance* function $d_S : G \times G \to \mathbb{R}_+$ defined by setting, for all $g, h \in G$,

$$d_S(g, h) = \ell_S(g^{-1} h).$$

Note that, denoting by $d_{Cay(G,S)}$ the *geodesic distance* in the Cayley graph $\mathcal{G}$ of G w.r. to S and identifying a group element in G with the corresponding vertex in $Cay(G,S)$, one has

$$d_S(g,h) = d_{Cay(G,S)}(g,h)$$

for all $g,h \in G$.

For $g \in G$ and $r \in \mathbb{N}$ we denote by $B(g;r) = \{h \in G : d_S(g,h) \leq r\}$ the *ball* of *radius* r centered at g.

In the classical case G is the lattice $\mathbb{Z}^2$ of integer points of Euclidean plane. In von Neumann's terminology [vN2] the generating system for $\mathbb{Z}^2$ is $S_{vN} = \{(1,0),(-1,0),(0,1),(0,-1)\}$ while in Moore's [Moo], $S_M = S_{vN} \cup \{(1,1),(1,-1),(-1,1),(-1,-1)\}$. Note that the corresponding Cayley graphs induce a tessellation of the Euclidean plane into squares; in the classical framework one considers these squares, called *cells*, rather than the vertices of these squares (i.e. the vertices of the graphs): this point of view is clearly equivalent as these graphs are isomorphic to their dual graphs (that is the graphs whose vertices are the cells, and two cells are neighbour if they share a common edge).
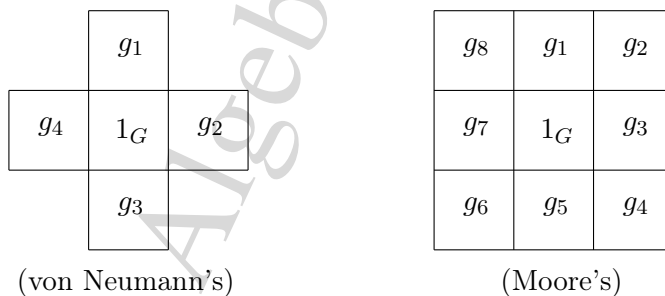

| (von Neumann's) | (Moore's) |

Figure 1. The ball $B(1_G;1)$ of radius one in $\mathbb{Z}^2$.

1.2. Cellular automata

The notion of a cellular automaton goes back to J. von Neumann [vN2] and S. Ulam [U].

Let G be a finitely generated group, called the *universe*, and let A be a set, called the *alphabet* or the *set of states*. A *configuration* is a map $x: G \to A$. The space A^G of all configurations is equipped with the right action of G defined by $G \times A^G \ni (g,x) \mapsto x^g \in A^G$, where $x^g(g') = x(gg')$ for all $g' \in G$.

One defines a distance function $d : A^G \times A^G \to \mathbb{R}_+$ by setting for $x, y \in A^G$

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{n} & \text{where } n = \max\{r \in \mathbb{N} : x|_{B(1_G;r)} \equiv y|_{B(1_G;r)}\}. \end{cases}$$

When A is finite, the configuration space A^G becomes a *compact* totally disconnected metric space.

A *cellular automaton* over G on the alphabet A is a map $\tau \colon A^G \to A^G$ satisfying the following condition: there exists a finite subset $M \subset G$ and a map $\mu \colon A^M \to A$ such that

$$\tau(x)(g) = \mu\left(x(g_1 g), x(g_2 g), \ldots, x(g_t g)\right) \equiv \mu(x^g|_M) \qquad (1.1)$$

for all $x \in A^G, g \in G$, where $x^g|_M$ denotes the restriction of x^g to M and $\{g_1, g_2, \ldots, g_t\} = M$. Such a set M is called a *memory set* and μ is called a *local defining map* for τ.

Note that, when G is finitely generated, say by a finite symmetric generating system S, it is not restrictive to choose as a memory set for τ, a ball $B(1_G; r)$ of radius $r \in \mathbb{N}$ centered at 1_G w.r. to the distance d_S induced by S.
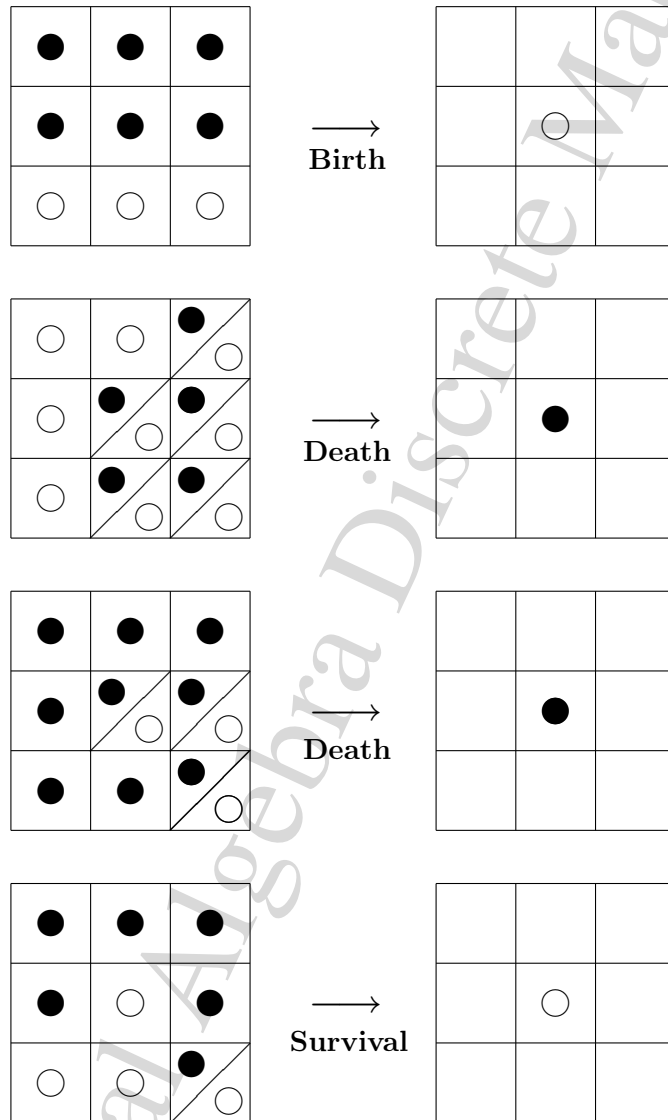
Let us note that every cellular automaton $\tau \colon A^G \to A^G$ is $G-$*equivariant*, i.e., it satisfies $\tau(x^g) = \tau(x)^g$ for all $g \in G$ and $x \in A^G$.

The following fact characterizes the "locality" of cellular automata from a topological point of view.

Theorem 1.1 (Hedlund). Suppose that A is finite. A map $\tau : A^G \to A^G$ is a cellular automaton if and only if τ is (uniformly) continuous and $G-$equivariant.

1.3. An example: Conway's game of life

In order to give the reader an explicit example of a cellular automaton we present here the *Game of life* of John H. Conway.

$\bullet = 0 =$ absence of life, $\bigcirc = 1 =$ presence of life.

Figure 2. The local rule for Conway's Game of life

The set of states is $A = \{0, 1\}$; state 0 corresponds to *absence of life* while state 1 indicates *life*; therefore passing from 0 to 1 can be interpreted as *birth*, while passing from 1 to 0 corresponds to *death*.

The universe is $\mathbb{Z}^2$ with Moore's system of generators (cfr. Figure 1), so that the ball $B((0,0); 1)$ consists of the 9 points $(x_1, x_2) \in \mathbb{Z}^2$ with $|x_1| \leq 1$ and $|x_2| \leq 1$.

The local map is defined as follows:

$$f(a_{(0,0)}, a_{(1,0)}, \ldots, a_{(-1,-1)}) = \begin{cases} 1 & \text{if} \begin{cases} \text{either } \sum_{s \in S} a_s = 3 \\ \text{or } \sum_{s \in S} a_s = 2 \text{ and } a_{(0,0)} = 1 \end{cases} \\ 0 & \text{otherwise.} \end{cases}$$

In other words [BCG, Chapter 25] for the transition map one has:

- **Birth:** a cell that is *dead* at time t becomes *live* at time $t + 1$ if and only if *three* of its neighbours are *live* at time t.

- **Survival:** a cell that is *live* at time t will remain *live* at $t + 1$ if and only if it has just *two or three live* neighbours at time t.

- **Death by overcrowding:** a cell that is *live* at time t and has *four or more* of its neighbours *live* at time t will be *dead* at time $t + 1$: food will not suffice.

- **Death by isolation:** a cell that has *at most one live* neighbour at time t becomes depressed and will be *dead* at time $t + 1$.

1.4. Amenability and entropy

Let G be a countable group (e.g. a finitely generated group) and let $\mathcal{P}(G)$ denote the set of all subsets of G. The group G is said to be *amenable* if there exists a *right-invariant mean* that is a map $\mu \colon \mathcal{P}(G) \to [0, 1]$ satisfying the following conditions:

(1) $\mu(G) = 1$ (*normalization*);

(2) $\mu(A \cup B) = \mu(A) + \mu(B)$ for all $A, B \in \mathcal{P}(G)$ such that $A \cap B = \varnothing$ (*finite additivity*);

(3) $\mu(Ag) = \mu(A)$ for all $g \in G$ and $A \in \mathcal{P}(G)$ (*right-invariance*).

We mention that if G is amenable, namely such a right-invariant mean exists, then also left-invariant and in fact even bi-invariant means do exist. The class of amenable groups includes, in particular, finite groups, solvable groups and finitely–generated groups of subexponential growth. It is closed under the operations of taking subgroups, taking factors, taking extensions and taking directed unions. The free group F_2 on two generators and therefore all groups containing non-abelian free subgroups are non-amenable. We mention that there exist periodic (and thus with no free subgroups) non amenable groups: the first examples are due to Ol'shanski [Ol]; later Adyan [Ady] showed that also the free Burnside groups $B(m, n) = \langle s_1, s_2, \ldots, s_m : w^n \rangle$ of rank $m \geq 2$ and odd *exponent* $n \geq 665$ are non-amenable. We mention that in both cases the non-amenability is detected by applying Grigorchuk's cogrowth criterion for

amenability.Recently Olshanskii and Sapir [OlS] found examples of non-amenable groups with no free subgroups in the class of *finitely presented* groups.

It follows from a result of Følner [Føl] that a countable group G is amenable if and only if it admits a *Følner sequence*, i.e., a sequence $(\Omega_n)_{n \in \mathbb{N}}$ of non-empty finite subsets of G such that

$$\lim_{n \to \infty} \frac{|\partial_E(\Omega_n)|}{|\Omega_n|} = 0 \quad \text{for all finite subsets } E \subset G, \qquad (1.2)$$

where $\partial_E(\Omega) = \Omega \triangle E\Omega$ is the symmetric difference of Ω and $E\Omega = \{e\omega : e \in E, \omega \in \Omega\}$.

For instance, for $G = \mathbb{Z}^2$ one can take as Følner sets the squares $\Omega_n = [-n, n] \times [-n, n]$.

Suppose that G is amenable and fix a Følner sequence $(\Omega_n)_{n \in \mathbb{N}}$. For a subset $X \subset A^G$ of configurations, in order to measure its "size" inside A^G, one introduces an *asymptotic density*, called *entropy*, as follows:

$$\text{ent}_{(\Omega_n)}(X) = \limsup_{n \to \infty} \frac{\log_{|A|} |X_{\Omega_n}|}{|\Omega_n|} \qquad (1.3)$$

where, for any subset $\Omega \subset G$

$$X_\Omega = \{x|_\Omega : x \in X\}$$

denotes the set of restrictions to Ω of all configurations in X.

Note that if X is $G-$invariant, then the above liminf in (1.3) is in fact a limit and does not depend on the particular Følner sequence $(\Omega_n)_{n \in \mathbb{N}}$; this follows from a well known result of Ornstein and Weiss in [OrW].

One clearly has $\text{ent}(A^G) = 1$ and $\text{ent}(X) \leq \text{ent}(Y)$ if $X \subset Y$.

1.5. Pre-injectivity, Gardens of Eden and entropy

Let Y be a set. Following a terminology introduced by Gromov [Gro1], one says that a map $f \colon A^G \to Y$ is *pre-injective* if the equality $f(x) = f(x')$ implies $x = x'$ for all $x, x' \in A^G$ such that $\{g \in G : x(g) \neq x'(g)\}$ is finite. Clearly, every injective map $A^G \to Y$ is pre-injective but the converse is not true in general.

Given a cellular automaton $\tau \colon A^G \to A^G$, one often speaks of τ as being time: if $x \in A^G$ is the configuration of the universe at time t, then $\tau[x]$ is the configuration of the universe at time $t + 1$. An *initial configuration* is a configuration at time $t = 0$. A configuration x not in the image of τ, namely $x \in A^G \setminus \tau[A^G]$, is called a *Garden of Eden configuration*, briefly a *GOE configuration*, this biblical terminology being

motivated by the fact that GOE configurations may only appear as initial configurations.

The following theorem is called the *Garden of Eden Theorem* because it gives a necessary and sufficient condition for the surjectivity of certain cellular automata. It was first proved by Moore and Myhill for $G = \mathbb{Z}^2$. Then it was extended to groups of sub-exponential growth by Machì and Mignosi [MaMi] and to general amenable groups by Ceccherini-Silberstein, Machì and Scarabotti [CMS].

Theorem 1.2. Let G be a countable amenable group and let A be a finite set. Let $\tau \colon A^G \to A^G$ be a cellular automaton. The following are equivalent:
- τ is surjective (i.e. there are no GOE configurations);
- τ is pre-injective;
- τ preserves the entropy: $\mathrm{ent}(\tau[A^G]) = 1$.

As Conway's game of life is concerned, we have that this cellular automaton is clearly not preinjective (the constant dead configuration and the configuration with only one live cell have the same image) and by the previous theorem it is not surjective either.

In the non-amenable setting the above theorem fails to hold in general. Let G be any group containing the free group F_2. In [CMS] it is presented the construction of two cellular automata over G such that: the first one is pre-injective not surjective, the second one is surjective not pre-injective.

It is not known whether the GOE theorem only holds for amenable groups or not: clearly, if this is the case, a new characterization of amenability would derive.

2. Surjunctivity and sofic groups

2.1. Surjunctivity

A group G is *surjunctive* (a terminology due to Gottschalk [Got]) if, given any finite alphabet A, any *injective* cellular automaton $\tau \colon A^G \to A^G$ is *surjective*. In other words, uniqueness implies existence for solutions of the equation $y = \tau(x)$. This property is reminiscent of several other classes of mathematical objects for which injective endomorphisms are always surjective (finite sets, finite-dimensional vector spaces, algebraic varieties, co-Hopfian groups, etc.).

Recall that a group G is said to be *residually finite* if for every element $g \neq 1_G$ in G there exist a finite group F and a homomorphism $h \colon G \to F$ such that $h(g) \neq 1_F$. This amounts to saying that the intersection of all subgroups of finite index of G is reduced to the identity element. The

class of residually finite groups is quite large. For instance, every finitely generated subgroup of $GL_n(\mathbb{C})$ is residually finite. However, there are finitely generated amenable groups, and even finitely generated solvable groups, which are not residually finite.

Lawton proved that residually finite groups are surjunctive.

From Theorem 1.2 one immediately deduces that amenable groups are surjunctive.

2.2. Sofic groups

Let S be a set. A *S-labeled graph* is a triple (Q, E, λ), where Q is a set, E is a symmetric subset of $Q \times Q$ and λ is a map from E to S. We shall view every subgraph of a labeled graph as a labeled graph in the obvious way.

Let (Q, E, λ) and (Q', E', λ') be S-labeled graphs. Two vertices $q \in Q$ and $q' \in Q'$ are said to be *r-equivalent* and we write $q \sim_r q'$ if the balls $B(q, r)$ and $B(q', r)$ are isomorphic as labeled graphs (i.e. there is a bijection $\varphi : B(q, r) \to B(q', r)$ sending q to q' such that $(q_1, q_2) \in E \cap (B(q, r) \times B(q, r))$ if and only if $(\varphi(q_1), \varphi(q_2)) \in E' \cap (B(q', r) \times B(q', r))$ and $\lambda(q_1, q_2) = \lambda'(\varphi(q_1), \varphi(q_2))$.

Let G be a finitely generated group and let S be a finite symmetric generating subset of G. The group G is said to be *sofic* if for all $\varepsilon > 0$ and $r \in \mathbb{N}$ there exists a finite S-labeled graph (Q, E, λ) such that the set $Q(r) \subset Q$ defined by $Q(r) = \{q \in Q : q \sim_r 1_G\}$ (here 1_G is considered as a vertex in the Cayley graph $Cay(G, S)$) satisfies

$$|Q(r)| \geq (1 - \varepsilon)|Q| \tag{2.1}$$

where $|\cdot|$ denotes cardinality.

It can be shown (see [Wei]) that this definition does not depend on the choice of S and that it can be extended as follows. A (not necessarily finitely generated) group G is said to be *sofic* if all of its finitely generated subgroups are sofic.

Sofic groups were introduced by M. Gromov in [Gro1] under the name of *initially amenable* groups. The sofic terminology is due to B. Weiss [Wei] The class of sofic groups contains, in particular, all residually amenable groups and it is closed under direct products, free products, taking subgroups and extensions by amenable groups [ES2].

Theorem 2.1 ([Wei]). *Sofic groups are surjunctive. In other words, given a finite set A and a sofic group G, any injective cellular automaton $\tau : A^G \to A^G$ is automatically surjective.*

We end this section by mentioning that there is no known example of a non-surjunctive group nor even of a non-sofic group up to now.

3.   Linear cellular automata

3.1.   Linear cellular automata and linear subshifts

A *linear cellular automaton* is a cellular automaton $\tau\colon V^G \to V^G$, where the alphabet V is a vector space over some field $\mathbb{K}$ and τ is linear.

As an example, let G be a group, S a non-empty finite (not necessarily symmetric) subset of G and V a vector space over a field $\mathbb{K}$ whose characteristic is not equal to the cardinality of S.

The *Laplace operator* associated with S is the linear cellular automaton $L_S\colon V^G \to V^G$ defined by

$$L_S(x)(g) = x(g) - \frac{1}{|S|}\sum_{s\in S} x(gs)$$

for all $x \in V^G$ and $g \in G$.

Note that L_S is never injective for $V \neq 0$ since all constant functions $G \to V$ are in the kernel of L_S.

Note that V^G (equipped with the Tychonov product topology where V is endowed with the discrete topology) is no longer a compact space so that many topological arguments based on compactness need to be obtained with an alternative method. As an example, the following facts, which are almost trivial in the classical (finite alphabet case) framework, need an appropriate proof.

Lemma 3.1 ([CeC1]). Let G be a countable group and let V be a finite-dimensional vector space over a field $\mathbb{K}$. Let $\tau\colon V^G \to V^G$ be a linear cellular automaton. Then $\tau(V^G)$ is a closed subspace of V^G.

A subset $X \subset V^G$ is called a *linear subshift* if it is a closed $G-$invariant vector subspace of V^G.

We extend our definition of a linear cellular automaton to linear subshifts in the following way.

Let $X, Y \subset V^G$ be linear subshifts. A map $\tau\colon X \to Y$ is called a *linear cellular automaton* if it is the restriction $\tau = \overline{\tau}|_X$ of a linear cellular automaton $\overline{\tau}\colon V^G \to V^G$.

Theorem 3.2 ([CeC2]). Let G be a countable group and let V be a finite-dimensional vector space over a field $\mathbb{K}$. Let $X, Y \subset V^G$ be linear subshifts and suppose that $\tau\colon X \to Y$ is a bijective cellular automaton. Then the inverse map $\tau^{-1}\colon Y \to X$ is also a linear cellular automaton.

3.2. Mean dimension

For linear cellular automata the following notion of mean dimension plays the role of entropy used in the classical framework, namely, for cellular automata with a finite alphabet.

Let G be a countable amenable group and V a finite-dimensional vector space over a field $\mathbb{K}$. Fix a Følner sequence $(\Omega_n)_{n \in \mathbb{N}}$ for G.

Given a vector subspace X of V^G and a subset $\Omega \subset G$, we denote by $X_\Omega \{x|_\Omega : x \in X\}$ the projection of X on V^Ω, where, as above, $x|_\Omega$ denotes the restriction of x to Ω.

The *mean dimension* of X (with respect to the Følner sequence $(\Omega_n)_{n \in \mathbb{N}}$) is the non–negative number

$$\mathrm{mdim}(X) = \liminf_{n \to \infty} \frac{\dim(X_{\Omega_n})}{|\Omega_n|}. \tag{3.1}$$

Note that it immediately follows from this definition that $\mathrm{mdim}(V^G) = \dim(V)$ and that $\mathrm{mdim}(X) \leq \dim(V)$ for every vector subspace X of V^G. More generally, one has $\mathrm{mdim}(X) \leq \mathrm{mdim}(Y)$ if X and Y are vector subspaces of V^G such that $X \subset Y$.

Again, if X is $G-$invariant, the result of Ornstein and Weiss [OrW] implies that in (3.1) we have a limit and that it does not depend on the particular choice of the Følner sequence $(\Omega_n)_{n \in \mathbb{N}}$ for G.

3.3. GOE theorem and L-surjunctivity for linear cellular automata

The linear analogue of the Garden of Eden Theorem for linear cellular automata states as follows.

Theorem 3.3 ([CeC1]). Let V be a finite-dimensional vector space over a field $\mathbb{K}$ and let G be a countable amenable group. Let $\tau \colon V^G \to V^G$ be a linear cellular automaton. Then the following are equivalent.
- τ is surjective (i.e. there are no GOE configurations);
- τ is pre-injective;
- τ preserves the mean dimension: $\mathrm{mdim}(\tau[V^G]) = \dim(V)$.

Also, say that a group G is $L-surjunctive$ if for any field $\mathbb{K}$ and finite dimensional vector space V over $\mathbb{K}$, any injective linear cellular automaton $\tau \colon V^G \to V^G$ is automatically surjective.

Theorem 3.4 ([CeC2]). Sofic groups are $L-$surjunctive.

4. Applications to ring theory

Let G be a group and $\mathbb{K}$ a field. We denote by $\mathbb{K}[G]$ the *group ring* of G with coefficients in $\mathbb{K}$, that is, the vector space over $\mathbb{K}$ with basis the elements of G, endowed with the bilinear product induced by multiplication in G.

4.1. Zero-divisors in group rings

Recall that, given a ring R, an element $a \in R$ is a (proper)*zero-divisor* if $a \neq 0$ and there exists $b \in R$ such that either $ab = 0$ or $ba = 0$.

If a group G has torsion, that is it contains an element $g_0 \neq 1_G$ of finite order, say $g_0^n = 1_G$, then

$$(1_G - g_0)(1_G + g_0 + g_0^2 + \cdots + g_0^{n-1}) = (1_G + g_0 + g_0^2 + \cdots + g_0^{n-1})(1_G - g_0) = 0,$$

thus showing that $\mathbb{K}[G]$ has proper zero-divisors.

A famous long standing open problem in the theory of group rings is *Kaplansky zero-divisor problem* [Kap1] which asks whether $\mathbb{K}[G]$ has no proper zero-divisors whenever G is torsion-free.

The zero-divisor problem is known to have an affirmative answer for the class of *unique-product* (briefly *u-p*) groups (a group G is u-p, [Pas1], if given two non-empty finite subsets $A, B \subset G$ there exists an element $g = g(A, B) \in G$ which can be uniquely expressed as a product $g = ab$ with $a \in A$ and $b \in B$); indeed if $\alpha = \sum_{a \in A} \lambda_a a$ and $\beta = \sum_{b \in B} \lambda'_b b \in \mathbb{K}[G]$ with non-empty finite supports A and $B \subset G$, respectively, then their product $\alpha\beta$ does not vanish at $g(A, B)$.

The class of u-p groups contains the class of *ordered* groups (a group G is ordered, [Pas1], if it admits a total order $\leq$ such that $g_1 \leq g_2$ implies $gg_1 \leq gg_2$ for all $g, g_1, g_2 \in G$); indeed we can take $g(A, B) = \min Ab_m$ where $b_m = \min B$.

In the class of ordered groups one finds the free groups, the free abelian groups, the fundamental groups of surfaces and the braid groups.

In the setting of linear cellular automata we have the following result which is a reformulation of Kaplansky zero-divisor problem.

Theorem 4.1 ([CeC1]). A group ring $\mathbb{K}[G]$ has no zero-divisors if and only if every linear cellular automaton $\tau \colon \mathbb{K}^G \to \mathbb{K}^G$ is pre-injective.

4.2. Direct finiteness of group rings

Recall that a ring R with identity element 1_R is said to be *directly finite* if one-sided inverses in R are also two-sided inverses, i.e., $ab = 1_R$ implies $ba = 1_R$ for $a, b \in R$. The ring R is said to be *stably finite* if the ring

$M_d(R)$ of $d \times d$ matrices with coefficients in R is directly finite for all integer $d \geq 1$.

Commutative rings and finite rings are obviously directly finite. Also observe that if elements a and b of a ring R satisfy $ab = 1_R$ then $(ba)^2 = ba$, that is, ba is an idempotent. Therefore if the only idempotents of R are 0_R and 1_R (e.g. if R has no proper zero-divisors) then R is directly finite. The class of directly finite rings is closed under taking subrings, direct products, and direct limits. The ring of endomorphisms of an infinite-dimensional vector space yields an example of a ring which is not directly finite.

Kaplansky [Kap2] observed that, for any group G and any field $\mathbb{K}$ of characteristic 0, the group ring $\mathbb{K}[G]$ is stably finite (see [Mon] and [Pas2] for detailed proofs in the case $\mathbb{K} = \mathbb{C}$) and asked whether this property remains true for fields of characteristic $p > 0$. We have that this holds for L-surjunctive groups.

Theorem 4.2 ([CeC2]). A group G is L-surjunctive if and only if the group ring $\mathbb{K}[G]$ is stably finite for any field $\mathbb{K}$.

As an immediate consequence of Theorems 3.4 and 4.2, we deduce the following result which has been previously established by G. Elek and A. Szabó [ES1] using different methods (their proof involves embeddings of the group rings into continuous von Neumann regular rings).

Corollary 4.3 ([ES1]). Let G be a sofic group and $\mathbb{K}$ any field. Then the group ring $\mathbb{K}[G]$ is stably finite. In particular, $\mathbb{K}[G]$ is directly finite.

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