

Isolated and nilpotent subsemigroups in the variants of \mathcal{IS}_n

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ABSTRACT. All isolated, completely isolated, and nilpotent subsemigroups in the semigroup \mathcal{IS}_n of all injective partial transformations of an n -element set, considered as a semigroup with a sandwich multiplication are described.

Introduction and main definitions

In the monograph [5] Ljapin proposed some constructions for the semigroups, a certain modification of which is the following. Let S be a semigroup. For a fixed $a \in S$ define an operation $*_a$ via $x *_a y = xay$, $x, y \in S$. Obviously, this multiplication $*_a$ is associative, therefore the set S with respect to this operation is the semigroup which is called *the semigroup S with a sandwich operation* or the *variant* of S and is denoted by $(S, *_a)$.

The variants of the classical transformation semigroups, \mathcal{IS}_n , \mathcal{T}_n , \mathcal{PT}_n , are interesting examples of this construction. In [7], [8], and [9], criteria of isomorphisms of these variants are detailed, and some of their properties are described.

Recall, a subsemigroup T of S is called *isolated*, provided that $x^k \in T$ for some $k \in \mathbb{N}$ implies $x \in S$ for all $x, y \in S$; T is called *completely isolated*, provided that $xy \in T$ implies $x \in T$ or $y \in T$ for all $x, y \in S$. Note that every completely isolated subsemigroup is isolated, while the contrary does not hold true. Isolated and completely isolated subsemigroups of the variants of \mathcal{T}_n are described in [6].

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A semigroup S with a zero 0 is called *nilpotent of the nilpotency degree* $k \geq 1$, provided that $x_1 x_2 \cdots x_k = 0$ for all $x_1, x_2, \dots, x_k \in S$ and there exist $y_1, y_2, \dots, y_{k-1} \in S$ such that $y_1 y_2 \cdots y_{k-1} \neq 0$. Denote $n(S) = k$. It is known[1], that a finite semigroup S is nilpotent if and only if each element of S is nilpotent, that is, for each $x \in S$ there exist $k \in \mathbb{N}$ such that $x^k = 0$ and $x^{k-1} \neq 0$.

In this paper we describe completely isolated, isolated and nilpotent subsemigroups of the variants of \mathcal{IS}_n , the inverse symmetric semigroup of all partial injective transformations of the set $N = \{1, 2, \dots, n\}$.

In particular, in section 1 we study isolated and completely isolated subsemigroups, and we produce the full description of nilpotent subsemigroups of the variants of \mathcal{IS}_n with respect to a natural zero (a nowhere defined map) in section 2. In the proofs we use the technique presented in [4].

For an arbitrary $\beta \in \mathcal{IS}_n$ denote $\text{dom}(\beta)$ and $\text{im}(\beta)$ the domain and image of β , respectively. The value $|\text{dom}(\beta)| = |\text{im}(\beta)|$ is called the range of β and is denoted by $\text{rank}(\beta)$.

It is proved in [8] that semigroups $(\mathcal{IS}_n, *_\alpha)$ and $(\mathcal{IS}_n, *_\beta)$ are isomorphic if and only if $\pi\alpha = \beta\tau$ for some permutations $\pi, \tau \in \mathcal{S}_n$. Note if α is a permutation then $(\mathcal{IS}_n, *_\alpha)$ is isomorphic to \mathcal{IS}_n , and the subsemigroup construction of the latter is studied in details in, for example,[2], [3]. Therefore, without loss of generality, we may assume that the sandwich element α is a non-identity idempotent in \mathcal{IS}_n .

Now let us fix an idempotent $\alpha \in \mathcal{IS}_n$. Set $(\mathcal{IS}_n, *) = (\mathcal{IS}_n, *_\alpha)$, $\text{dom}(\alpha) = A$, $\text{rank}(\alpha) = l$, $l < n$, and let α , as an element of \mathcal{IS}_n , be defined on A identically.

Fix some $z \in N \setminus A$ and for arbitrary pairwise different elements $x_1, \dots, x_k \in A$ define the following partial permutations:

$$(x_1, \dots, x_k)(x) = \begin{cases} x_{i+1}, & \text{if } x = x_i, \quad i < k, \\ x_1, & \text{if } x = x_k, \\ x, & \text{if } x \in N \setminus \{x_1, \dots, x_k\} \end{cases}$$

and

$$[x_1, \dots, x_k](x) = \begin{cases} x_{i+1}, & \text{if } x = x_i, \quad i < k, \\ z, & \text{if } x = x_k, \\ x, & \text{if } x \in N \setminus \{x_1, \dots, x_k, z\}. \end{cases}$$

For any $\beta \in (\mathcal{IS}_n, *)$ we denote $\beta^{*s} = \underbrace{\beta * \beta * \dots * \beta}_s$.

Proposition 1 ([8]). *An element $\varepsilon \in \mathcal{IS}_n$ is an idempotent in $(\mathcal{IS}_n, *)$ if and only if ε is an idempotent in \mathcal{IS}_n and $\text{dom}(\varepsilon) \subset A$.*

Remark 2. For each $x \in A$ denote ε_x the idempotent of \mathcal{IS}_n such that $\text{dom}(\varepsilon_x) = A \setminus \{x\}$. Then any idempotent $\varepsilon \neq \alpha$ of $(\mathcal{IS}_n, *)$ may be factorized as

$$\varepsilon = \prod_{x \in A \setminus \text{dom}(\varepsilon)} \varepsilon_x = \prod_{x \in A \setminus \text{dom}(\varepsilon)}^* \varepsilon_x.$$

1. Isolated and completely isolated subsemigroups in $(\mathcal{IS}_n, *)$

Let S be a semigroup. For the idempotent e in S define

$$\sqrt{e} = \{x \in S : x^m = e \text{ for some } m > 0\}.$$

Proposition 3 ([6]). *If \sqrt{e} is a subsemigroup of S then \sqrt{e} is a minimal with respect to inclusion isolated subsemigroup containing e .*

Denote \mathcal{C}_A the set of those elements from \mathcal{IS}_n , which are one-to-one maps on A and are arbitrarily defined on $N \setminus A$. That is,

$$\mathcal{C}_A = \{\beta \in \mathcal{IS}_n \mid \beta(A) = A\}.$$

Theorem 4. *The only completely isolated subsemigroups of $(\mathcal{IS}_n, *)$ are \mathcal{C}_A , $(\mathcal{IS}_n, *) \setminus \mathcal{C}_A$ and $(\mathcal{IS}_n, *)$.*

Proof. It is clear that \mathcal{C}_A is a subsemigroup of $(\mathcal{IS}_n, *)$. Let $\beta * \gamma \in \mathcal{C}_A$ for some β, γ from $(\mathcal{IS}_n, *)$. Then $\text{dom}(\beta * \gamma) \supset A$, therefore $\beta(x) \in A$ for all $x \in A$. This implies $\beta \in \mathcal{C}_A$. Hence \mathcal{C}_A is a completely isolated subsemigroup. $(\mathcal{IS}_n, *) \setminus \mathcal{C}_A$ is also completely isolated as a complement to the completely isolated subsemigroup. Obviously, $(\mathcal{IS}_n, *)$ is completely isolated.

Conversely, let $T \subset (\mathcal{IS}_n, *)$ be a completely isolated subsemigroup. Assume that $T \cap \mathcal{C}_A \neq \emptyset$, $\beta \in \mathcal{C}_A \cap T$. Then $\beta^{*s} = \alpha$ for some $s \in \mathbb{N}$ hence $\alpha \in T$. However, for any γ from \mathcal{C}_A we have $\gamma^{*t} = \alpha$ for some $t \in \mathbb{N}$, hence $\gamma \in T$. Therefore

$$T \cap \mathcal{C}_A = \emptyset \quad \text{or} \quad \mathcal{C}_A \subset T. \quad (1)$$

Let $T \cap ((\mathcal{IS}_n, *) \setminus \mathcal{C}_A) \neq \emptyset$ and β belong to the intersection. Then T contains an idempotent ε as a power of β and $\text{rank}(\varepsilon) < l$. From remark 2 it follows that since T is completely isolated $\varepsilon_x \in T$ for some $x \in A$. But for any $y \in A$ we have $\varepsilon_x = (x, y) * \varepsilon_y * (x, y)$. So if $\mathcal{C}_A \subset T$ then T contains ε_y , as it is a semigroup. Otherwise T contains ε_y as it is completely isolated. Consequently, T contains all idempotents of ranks $< l$, and since some power of each element $\gamma \in (\mathcal{IS}_n, *) \setminus \mathcal{C}_A$ equals such idempotent, we get

$$((\mathcal{IS}_n, *) \setminus \mathcal{C}_A) \subset T \quad \text{or} \quad ((\mathcal{IS}_n, *) \setminus \mathcal{C}_A) \cap T = \emptyset. \quad (2)$$

Now the statement of the theorem follows from (1) and (2). \square

For every $x \in A$ denote

$$\mathcal{G}(x) = \{\lambda \in \mathcal{IS}_n : \lambda(x) \notin A \text{ and } \lambda(y) \in A \setminus \{x\} \text{ for all } y \in A \setminus \{x\}\}.$$

Observe, if $|A| = 1$ then $\mathcal{G}(x) = (\mathcal{IS}_n, *) \setminus \mathcal{C}_A$, $x \in A$.

Lemma 5. $\mathcal{G}(x) = \sqrt{\varepsilon_x}$.

Proof. It is clear that $\mathcal{G}(x)$ is a subsemigroup of $(\mathcal{IS}_n, *)$ and $\mathcal{G}(x) \subset \sqrt{\varepsilon_x}$. Further $\varepsilon_x \in \mathcal{G}(x)$ by definition of $\mathcal{G}(x)$.

Let $\lambda^{*k} \in \mathcal{G}(x)$ for some $k \in \mathbb{N}$. If $\lambda(x) \in A$ then whereas λ^{*k} is not one-to-one map on A there exists $y \in A \setminus \{x\}$ such that $\lambda(y) \in N \setminus A$. Then $\lambda^{*k}(y)$ is not defined for $k \geq 2$, which contradicts the definition of $\mathcal{G}(x)$. Now let $\lambda(x) \in N \setminus A$. Clearly, for $\lambda^{*k} \in \mathcal{G}(x)$ it is necessary $\lambda(y) \in A \setminus \{x\}$ for all $y \in A \setminus \{x\}$. Hence, $\lambda \in \mathcal{G}(x)$ and $\mathcal{G}(x)$ is isolated subsemigroup. Finally, proposition 3 completes the proof. \square

Theorem 6. (i) If $\text{rank}(\alpha) \geq 2$ then the only isolated subsemigroups of $(\mathcal{IS}_n, *)$ are $(\mathcal{IS}_n, *)$, \mathcal{C}_A , $(\mathcal{IS}_n, *) \setminus \mathcal{C}_A$ and $\mathcal{G}(x)$, $x \in A$.

(ii) If $\text{rank}(\alpha) = 1$ then the only isolated subsemigroups of $(\mathcal{IS}_n, *)$ are \mathcal{C}_A , $(\mathcal{IS}_n, *) \setminus \mathcal{C}_A$, $(\mathcal{IS}_n, *)$, in particular all of them are completely isolated.

Proof. By theorem 4 and lemma 5 all listed subsemigroups are isolated.

Let T be isolated subsemigroup of $(\mathcal{IS}_n, *)$. As in the proof of theorem 4 it can be shown that (1) holds.

Assume that $\mathcal{C}_A \subset T$. If $T \neq \mathcal{C}_A$ then there exists $\beta \in T \setminus \mathcal{C}_A$ and T contains some idempotent ε of rank $< l$. Denote $A \setminus \text{im}(\varepsilon) = \{a_1, \dots, a_k\}$ and consider elements $\lambda = [a_1, \dots, a_k]$ and $\mu = [a_k, \dots, a_1]$. From $\lambda^{*k} = \mu^{*k} = \varepsilon$ it follows that $\lambda, \mu \in T$ and $\lambda * \mu = \varepsilon_{a_k}$ belongs to T . However as $\mathcal{C}_A \subset T$, T contains all ε_x , $x \in A$, and hence it contains all idempotents of ranks $\leq l - 1$. Therefore in this case $T = (\mathcal{IS}_n, *)$.

Now assume that $\mathcal{C}_A \cap T = \emptyset$ and T contains an element β and some idempotent ε of rank $< l$ as a power of β . Consider the cases.

1) The case of $l = 1$.

Obviously, the power of any element from $(\mathcal{IS}_n, *) \setminus \mathcal{C}_A$ is a nowhere defined map, so $T = (\mathcal{IS}_n, *) \setminus \mathcal{C}_A$.

2) The case of $l \geq 2$.

If $\text{rank}(\varepsilon) \leq l - 2$ then by the analogous arguments produced above it can be shown that T contains at least two different idempotents ε_x

and ε_y , $x, y \in A$, $x \neq y$. We show that T contains all idempotents of rank $l - 1$, and hence, $T = (\mathcal{IS}_n, *) \setminus \mathcal{C}_A$. Indeed, let $z \in A$ and $z \neq x, y$. Consider element $\mu = (x, z) * [y]$. $\mu^{*2} = \varepsilon_y$, hence $\mu \in T$. Then $(\varepsilon_x * \mu)^{*2} = [x] * [y] * [z] \in T$. Let $\sigma = [x, y, z]$, $\tau = [z, y, x]$. We have $\sigma^{*3} = \tau^{*3} = [x] * [y] * [z]$, hence $\sigma, \tau \in T$ and finally $\sigma * \tau = \varepsilon_z \in T$.

Now let $\text{rank}(\varepsilon) = l - 1$. If ε is not the only idempotent in T then by the above reasoning $T = (\mathcal{IS}_n, *) \setminus \mathcal{C}_A$. If ε is the only idempotent in T then $T = \mathcal{G}(x)$.

The theorem is proved. \square

2. Nilpotent subsemigroups in $(\mathcal{IS}_n, *)$

For every positive integer k denote Nil_k the set of all nilpotent subsemigroups of $(\mathcal{IS}_n, *)$ of nilpotency degree $\leq k$. The set Nil_k is partially ordered with respect to inclusions in a natural way. Set $M = \overline{A}^{(1)} \cup A \cup \overline{A}^{(2)}$, where $\overline{A}^{(1)} = N \setminus A$, and $\overline{A}^{(2)}$ is a disjoint copy of $\overline{A}^{(1)}$. For every $x \in \overline{A}^{(1)}$ denote x' the corresponding element from $\overline{A}^{(2)}$.

Denote $\text{Ord}_k(M)$ the ordered set of all strict partial orders, Λ , on M which satisfy the following two conditions:

- (1) the cardinalities of chains of Λ are bounded by k ,
- (2) $\overline{A}^{(1)} \subseteq \min(\Lambda)$, $\overline{A}^{(2)} \subseteq \max(\Lambda)$, where $\min(\Lambda)$ and $\max(\Lambda)$ mean the sets of all minimal and maximal elements of order Λ respectively.

If $k \leq m$ then we have natural inclusions $\text{Nil}_k \hookrightarrow \text{Nil}_m$ and $\text{Ord}_k(M) \hookrightarrow \text{Ord}_m(M)$, which preserve the partial order. Therefore we can consider the ordered sets

$$\text{Nil} = \bigcup_k \text{Nil}_k \quad \text{and} \quad \text{Ord}(M) = \bigcup_k \text{Ord}_k(M).$$

For every partial order $\Lambda \in \text{Ord}(M)$ consider the set

$$\text{Mon}(\Lambda) = \{\beta \in \mathcal{IS}_n : \beta(x) \neq x \text{ and } (x, \beta(x)) \in \Lambda \text{ for all } x \in \text{dom}(\beta)\} \quad (3)$$

and for every subsemigroup $S \in \text{Nil}$ the relation

$$\begin{aligned} \Lambda_S = & \{(x, y) : x \in A \text{ and there exists } \beta \in S \text{ such that } \beta(x) = y\} \\ & \cup \{(x, y) : y \in A \text{ and there exists } \beta \in S \text{ such that } \beta(x) = y\} \\ & \cup \{(x, y) : x \in \overline{A}^{(1)}, y \in \overline{A}^{(2)} \text{ and there exists } \beta \in S \text{ such that } \beta(x) = y\}. \end{aligned} \quad (4)$$

Let $\beta \in \mathcal{IS}_n$ and $\text{rank}(\beta) = k$, $k \leq n$. Write β as

$$\beta = \begin{pmatrix} x_{11} & \dots & x_{1i_1} & x_{21} & \dots & x_{2i_2} & x_{31} & \dots & x_{3i_3} & x_{41} & \dots & x_{4i_4} \\ y_{11} & \dots & y_{1i_1} & y_{21} & \dots & y_{2i_2} & y_{31} & \dots & y_{3i_3} & y_{41} & \dots & y_{4i_4} \end{pmatrix}, \quad (5)$$

that is, $\text{dom}(\beta) = \{x_{11}, \dots, x_{4i_4}\}$ and $\beta(x_{ij}) = y_{ij}$ for all i, j , moreover

$$\begin{aligned} \{x_{11}, \dots, x_{1i_1}\} &\subset A, & \{y_{11}, \dots, y_{1i_1}\} &\subset A, \\ \{x_{21}, \dots, x_{2i_2}\} &\subset A, & \{y_{21}, \dots, y_{2i_2}\} &\subset N \setminus A, \\ \{x_{31}, \dots, x_{3i_3}\} &\subset N \setminus A, & \{y_{31}, \dots, y_{3i_3}\} &\subset A, \\ \{x_{41}, \dots, x_{4i_4}\} &\subset N \setminus A, & \{y_{41}, \dots, y_{4i_4}\} &\subset N \setminus A. \end{aligned}$$

Consider a map $f : (\mathcal{IS}_n, *) \rightarrow \mathcal{IS}(M)$ defined in the following way:

if β is given by (5) then $\text{dom}(f(\beta)) = \{x_{11}, \dots, x_{1i_1}, x_{21}, \dots, x_{4i_4}\}$ and

$$f(\beta) = \begin{pmatrix} x_{11} & \dots & x_{1i_1} & x_{21} & \dots & x_{2i_2} & x_{31} & \dots & x_{3i_3} & x_{41} & \dots & x_{4i_4} \\ y_{11} & \dots & y_{1i_1} & y'_{21} & \dots & y'_{2i_2} & y_{31} & \dots & y_{3i_3} & y'_{41} & \dots & y'_{4i_4} \end{pmatrix}.$$

Proposition 7. *The above map $f : (\mathcal{IS}_n, *) \rightarrow \mathcal{IS}(M)$ is a monomorphism, besides,*

$$\text{Im}(f) = \{\gamma \in \mathcal{IS}(M) : \text{dom}(\gamma) \subset \overline{A}^{(1)} \cup A, \text{im}(\gamma) \subset A \cup \overline{A}^{(2)}\}.$$

Proof. Clear from the definition. □

Hence, every nilpotent subsemigroup of $(\mathcal{IS}_n, *)$ is mapped by f to the corresponding nilpotent subsemigroup of $\mathcal{IS}(M)$. This allows one to apply the results from [4] for $\mathcal{IS}(M)$ to the semigroup $(\mathcal{IS}_n, *)$. In particular, by a word for word repetition of corresponding proofs from [4] one may prove propositions 8-10:

Proposition 8. (i) *For each $k \geq 1$ the map $\Lambda \mapsto \text{Mon}(\Lambda)$ is a homomorphism from the poset $\text{Ord}_k(M)$ to the poset Nil_k .*

(ii) *For every $k \geq 1$ the map $S \mapsto \Lambda_S$ is a homomorphism from the poset Nil_k to the poset $\text{Ord}_k(M)$.*

Proposition 9. *Let $n(S) = k$. Then $\Lambda_S \in \text{Ord}_k(M) \setminus \text{Ord}_{k-1}(M)$.*

Proposition 10. *Let $S \in \text{Nil}$ and $\Lambda \in \text{Ord}(M)$. Then*

(i) $\text{Mon}(\Lambda_S) \supset S$, $\Lambda_{\text{Mon}(\Lambda)} \subset \Lambda$;

(ii) $\text{Mon}(\Lambda_{\text{Mon}(\Lambda)}) = \text{Mon}(\Lambda)$;

(iii) $\Lambda_{\text{Mon}(\Lambda_S)} = \Lambda_S$.

From proposition 9 we derive

Corollary. *The nilpotency degree, $n(S)$, of any nilpotent subsemigroup $S \subset (\mathcal{IS}_n, *)$ does not exceed $\text{rank}(\alpha) + 2$, moreover the equality holds true if and only if $|N \setminus \text{im}(\alpha)| \geq 1$ and $\min(\Lambda_S) = \overline{A}^{(1)}$, $\max(\Lambda_S) = \overline{A}^{(2)}$. All maximal nilpotent subsemigroups are isomorphic.*

By an *ordered A-partition* of M into k non-empty blocks we mean the partition $M = M_1 \cup M_2 \cup \dots \cup M_k$ where $M_1 \supset \overline{A}^{(1)}$, $M_k \supset \overline{A}^{(2)}$, and the order of the blocks is also taken into account. With every ordered A -partition of $M = M_1 \cup M_2 \cup \dots \cup M_k$ we associate the set

$$\text{ord}(M_1, \dots, M_k) = \bigcup_{1 \leq i < j \leq k} M_i \times M_j \subset M \times M. \quad (6)$$

Lemma 11. *Let $k \leq |M|$ be fixed. Then*

- (i) *for every ordered A-partition $M = M_1 \cup M_2 \cup \dots \cup M_k$ the set $\text{ord}(M_1, \dots, M_k)$ is a maximal element in $\text{Ord}_k(M)$;*
- (ii) *different ordered A-partitions of M correspond to different elements in $\text{Ord}_k(M)$;*
- (iii) *each maximal element in $\text{Ord}_k(M)$ has the form $\text{ord}(M_1, \dots, M_k)$ for some ordered A-partition of $M = M_1 \cup M_2 \cup \dots \cup M_k$.*

Proof. Statements (i) and (ii) are proved analogously to lemma 7 from [4].

(iii) Let the order $\Lambda \in \text{Ord}_k(M)$ be fixed. Denote by M_1 the set of all minimal elements of the relation Λ . By definition $\overline{A}^{(1)} \subset M_1$, therefore $M_1 \neq \emptyset$. For every increasing (with respect to Λ) chain $x_1 < \dots < x_m$ of elements in $M \setminus M_1$ there exists $x_0 \in M$ such that $x_0 < x_1$. Hence, the cardinality of every increasing chain in $M \setminus M_1$ is bounded by $k - 1$. Denote by M_2 the set of all minimal elements in $M \setminus M_1$, by M_3 the set of all minimal elements in $M \setminus (M_1 \cup M_2)$ and so on. In k steps we get the partition $M = M_1 \cup M_2 \cup \dots \cup M_k$, for which $\overline{A}^{(1)} \subset M_1$. Observe that the elements of $\overline{A}^{(2)}$ are maximal by the definition of Λ . Hence set

$$\begin{aligned} \widetilde{M}_i &= M_i \setminus \overline{A}^{(2)}, \quad 1 \leq i \leq k - 1, \\ \widetilde{M}_k &= M_k \cup \overline{A}^{(2)} \end{aligned}$$

and get the new partition $M = \widetilde{M}_1 \cup \widetilde{M}_2 \cup \dots \cup \widetilde{M}_k$. Clearly, $\Lambda \subset \text{ord}(\widetilde{M}_1, \dots, \widetilde{M}_k)$ and from the maximality of Λ we have $\Lambda = \text{ord}(\widetilde{M}_1, \dots, \widetilde{M}_k)$. The lemma is proved. \square

For every ordered A -partition of $M = M_1 \cup M_2 \cup \dots \cup M_k$ denote

$$T(M_1, \dots, M_k) = \{\beta \in \mathcal{IS}_n : x \in M_i \text{ and } \beta(x) \in M_j \text{ imply } i < j \\ \text{for all } x \in \text{dom}(\beta)\}. \quad (7)$$

Lemma 12.

$$T(M_1, \dots, M_k) = \text{Mon}(\text{ord}(M_1, \dots, M_k)).$$

Proof. The assertion follows from definitions (3), (4), (6) and (7). \square

Theorem 13. (i) For every ordered A -partition of $M = M_1 \cup M_2 \cup \dots \cup M_k$ the semigroup $T(M_1, \dots, M_k)$ is maximal in Nil_k .

(ii) Different ordered A -partitions of M correspond to different subsemigroups in Nil_k .

(iii) Every maximal subsemigroup in Nil_k has the form $T(M_1, \dots, M_k)$ for some ordered A -partition of $M = M_1 \cup M_2 \cup \dots \cup M_k$.

Proof. From lemma 12 it follows that the set $T(M_1, \dots, M_k)$ is a subsemigroup in Nil_k . We show that

$$\Lambda_{T(M_1, \dots, M_k)} = \text{ord}(M_1, \dots, M_k). \quad (8)$$

From proposition 10(i) we have $\Lambda_{T(M_1, \dots, M_k)} \subset \text{ord}(M_1, \dots, M_k)$.

We prove the contrary inclusion. Let $(x, y) \in \text{ord}(M_1, \dots, M_k)$. Take $\beta \in \mathcal{IS}_n$ such that $\text{rank}(\beta) = 1$, $\text{dom}(\beta) = \{x\}$, $\text{im}(\beta) = \{y\}$. By definition $\beta \in \text{Mon}(\text{ord}(M_1, \dots, M_k))$, and using (4) we have $(x, y) \in \Lambda_{\text{Mon}(\text{ord}(M_1, \dots, M_k))}$. Hence $\Lambda_{T(M_1, \dots, M_k)} \supset \text{ord}(M_1, \dots, M_k)$ and the equality (8) is proved.

Now let $S \in \text{Nil}_k$ be such that $S \supset T(M_1, \dots, M_k)$. According to lemma 11 the order $\text{ord}(M_1, \dots, M_k)$ is a maximal element in $\text{Ord}_k(M)$, therefore by proposition 8(ii) we get $\Lambda_S = \text{ord}(M_1, \dots, M_k)$. Finally from proposition 10(i), lemma 12, and equality (8) it follows

$$S \subset \text{Mon}(\Lambda_S) = \text{Mon}(\text{ord}(M_1, \dots, M_k)) = T(M_1, \dots, M_k)$$

and the statement (i) is proved.

Statements (ii) and (iii) follow from proposition 8, lemma 11 and lemma 12. \square

Let S be maximal nilpotent subsemigroup of $(\mathcal{IS}_n, *)$ of nilpotency degree k and let $M = M_1 \cup M_2 \cup \dots \cup M_k$ be the ordered A -partition of $M = \overline{A}^{(1)} \cup A \cup \overline{A}^{(2)}$, which corresponds to the partial order Λ_S . We call the set $(|M_1|, |M_2|, \dots, |M_k|) \in \mathbb{N}^k$ the type of nilpotent subsemigroup S and denote it by $\text{type}(S)$. Set $(|M_1|, \dots, |M_k|)^\# = (|M_k|, \dots, |M_1|)$.

Proposition 14. *Let T_1 and T_2 be two maximal nilpotent subsemigroups of $(\mathcal{IS}_n, *)$ of nilpotency degree k . Then*

- (i) *if $k = 2$ then T_1 and T_2 are isomorphic if and only if $\text{type}(T_1) = \text{type}(T_2)$ or $\text{type}(T_1) = \text{type}(T_2)^\#$.*
- (ii) *if $k > 2$ then T_1 and T_2 are isomorphic if and only if $\text{type}(T_1) = \text{type}(T_2)$. T_1 and T_2 are anti-isomorphic if and only if $\text{type}(T_1) = \text{type}(T_2)^\#$.*

The proof follows from proposition 7 and corresponding statements about nilpotent subsemigroups of $\mathcal{IS}(M)$ from [3] (lemma 14.4 and theorem 14.1).

□

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