Algebra and Discrete Mathematics RESEARCH ARTICLE
Number 1. **(2006).** pp. 1 – 17

Strongly orthogonal and uniformly orthogonal many-placed operations

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Communicated by A. I. Kashu

Abstract. In [3] we have studied connection between orthogonal hypercubes and many-placed (d-ary) operations, have considered different types of orthogonality and their relationships. In this article we continue study of orthogonality of many-placed operations, considering special types of orthogonality such as strongly orthogonality and uniformly orthogonality. We introduce distinct types of strongly orthogonal sets and of uniformly orthogonal sets of d-ary operations, consider their properties and establish connections between them.

1. Introduction

In the article [3] it was established a connection between d-dimentional hypercubes of different types and many-placed (the same d-ary, polyadic or multary) operations. Distinct types of orthogonality of many-placed operations (of d-dimentional hypercubes) and relationship between them were considered. In this article we continue study of orthogonality of many-placed operations, in particular, we consider special types of orthogonality such that strongly orthogonality and uniformly orthogonality. We introduce distinct types of strongly orthogonal sets and of uniformly

Acknowledgment: The research described in this article was made possible in part by Award No. MM1-3040-CH-02 of the Moldovan Research and Development Association (MRDA) and the U.S. Civilian Research & Development Foundation for the Independent States of the Former Soviet Union (CRDF).

2000 Mathematics Subject Classification: *20N05, 20N15, 05B15.*

Key words and phrases: *hypercube, orthogonal hypercubes, d-ary operation, d-ary quasigroup, orthogonal d-ary operations, strongly orthogonal operations, uniformly orthogonal operations.*

orthogonal sets of many-placed (d-ary) operations and establish connections between them. In parallel, types of orthogonality are considered for sets of polynomial d-operations over a field and some examples of such sets are given.

Note, that taking into account the connection these results with d-dimentional hypercubes and with the results of the paper [3], we use the letter d for designation of an arity and the letter n is used for designation of an order of an operation.

2. Necessary notions and results

We recall some notations, concepts and results which are used in the article. At first remember the following denotes and notes from [2]. By x_i^j we will denote the sequence $x_i, x_{i+1}, \ldots, x_j$, $i \le j$. If $j < i$, then x_i^j is the empty sequence, $\overline{1,n} = \{1, 2, ..., n\}$. Let Q be a finite or an infinite set, $d \ge 1$ be a positive integer, and let Q^d denote the Cartesian power of the set Q.

A d-ary operation A (briefly, a d-operation) on a set Q is a mapping $A : Q^d \to Q$ defined by $A(x_1^d) \to x_{d+1}$, and in this case we write $A(x_1^d) = x_{d+1}$. Thus, an 1-ary (unary) operation is simply a mapping from Q into Q.

A d-groupoid (Q, A) *of order n* is a set Q with one d-ary operation A defined on Q, where $|Q| = n$.

A *d-ary quasigroup* is a d-groupoid such that in the equality

$$A(x_1^d) = x_{d+1}$$

each of d elements from x_1^{d+1} uniquely defines the $(d+1)$-th element. Usually a quasigroup d-operation A is itself considered as a d-quasigroup.

The d-operation E_i, $1 \le i \le d$, on Q with $E_i(x_1^d) = x_i$ is called *the i-th identity operation (or the i-th selector) of arity d.*

Let j be a fixed number, $0 \le j \le d-1$, $\{i_1, i_2, ..., i_j\} \subseteq \overline{1,d}$, $(a_{i_1}, a_{i_2}, ..., a_{i_j}) \in Q^j$.

By I_j we denote the set of all $C_d^j \cdot \mid Q \mid^j$, $2j$-tuples

$$\bar{a} = (i_1, i_2, ..., i_j; a_{i_1}, a_{i_2}, ..., a_{i_j})$$

when the set $\{i_1, i_2, ..., i_j\}$ runs trough over all C_d^j, j-subsets of $\overline{1,d}$ and $(a_{i_1}, a_{i_2}, ..., a_{i_j})$ runs trough all $\mid Q \mid^j$, j-tuples of elements of Q, that is

$$I_j = \{(i_1, i_2, ..., i_j; a_{i_1}, a_{i_2}, ..., a_{i_j}) \mid \{i_1, i_2, ..., i_j\} \subseteq \overline{1,d}, (a_{i_1}, a_{i_2}, ..., a_{i_j}) \in Q^j\},$$

if $j > 0$ and put $I_0 = \emptyset$ (the empty set).

Let A be a d-ary operation, $\bar{a} = (i_1, i_2, ..., i_j; a_{i_1}, a_{i_2}, ..., a_{i_j}) \in I_j$. Changing j variables $x_{i_1}, x_{i_2}, ..., x_{i_j}$ in A on fixed elements $a_{i_1}, a_{i_2}, ..., a_{i_j}$ of Q respectively we obtain a new operation

$$A(x_1^{i_1-1}, a_{i_1}, x_{i_1+1}^{i_2-1}, a_{i_2}, \ldots, x_{i_j-k}^{i_j-1}, a_{i_j}, x_{i_j+1}^{d}) =$$

$$A_{\bar{a}}(x_1^{i_1-1}, x_{i_1+1}^{i_2-1}, \ldots, x_{i_j-k}^{i_j-1}, x_{i_j+1}^{d}) = B_{\bar{a}}(y_1^{d-j}),$$

if we rename the remaining $d-j$ variables in the following way:

$$(x_1^{i_1-1}, x_{i_1+1}^{i_2-1}, \ldots, x_{i_j+1}^{d}) = (y_1^{i_1-1}, y_{i_1}^{i_2-1}, ..., y_{i_j}^{d}) = (y_1^{d-j}).$$

Then $B_{\bar{a}}$ is a $(d-j)$-ary operation, which is called *the* $(d-j)$*-ary retract (shortly, the* $(d-j)$*-retract) of* A, defined by the $2j$-tuple $\bar{a} \in I_j$. If $\bar{a} \in I_0 = \emptyset$, then $B_{\bar{a}} = A$.

Recall (see [4],[5]) that for $d \geq 2$ a *d-dimentional hypercube (briefly, a d-hypercube) of order n* is a $\underbrace{n \times n \times \cdots \times n}_{d}$ array with n^d points based upon n distinct symbols. Such a d-hypercube has *type* j with $0 \leq j \leq d-1$ if, whenever any j of the d coordinates are fixed, each of the n symbols appears n^{d-j-1} times in that subarray.

A hypercube is a generalization of a *latin square*, which in the case of a square of *order* n, is a $n \times n$ array in which n distinct symbols are arranged so that each symbol occurs once in each row and each column. A latin square is a 2-dimensional hypercube of type 1.

Some d-ary algebraic operation A_H on a set Q of type j corresponds to a d-hupercube H of type j based on the set Q and conversely [3].

By Proposition 1 of [3] a d-hypercube (a d-operation A_H) defined on a set Q of order n has type j with $0 \leq j \leq d-1$ if and only if for each $(d-j)$-retract $B_{\bar{a}}(y_1^{d-j}), \bar{a} \in I_j$, of the corresponding d-operation A_H, the equation $B_{\bar{a}}(y_1^{d-j}) = b$ has exactly n^{d-j-1} solutions for each $b \in Q$.

A d-hypercube H (a d-operation A_H) has type $j = d-1$ if and only if the d-operation A_H is a d-quasigroup ([3], Corollary 1).

Two d-hypercubes H_1 and H_2 of order n are *orthogonal* if when superimposed, each of the n^2 ordered pairs appears n^{d-2} times, and a set of $t \geq 2$, d-hypercubes is *orthogonal* if every pair of distinct d-hypercubes is orthogonal; see [4],[5].

Two d-operations A and B of order n defined on a set Q are said to be orthogonal if the pair of equations $A(x_1^d) = a$ and $B(x_1^d) = b$ has exactly n^{d-2} solutions for any elements $a, b \in Q$ ([3], Definition 4).

A set $\Sigma = \{A_1, A_2, \ldots, A_t\}$ of d-operations with $t \geq 2$ is called orthogonal if every pair of distinct d-operations from Σ is orthogonal ([3], Definition 5).

Two d-hypercubes H_1 and H_2 are orthogonal if and only if the respective d-operations A_{H_1} and A_{H_2} are orthogonal. A set of (pairwise) orthogonal d-operations corresponds to a set of (pairwise) orthogonal d-hypercubes.

In [3] this notion of orthogonality was generalized in the following way.

Definition 1 ([3]). *A k-tuple $< A_1, A_2, \dots, A_k >$, $1 \le k \le d$, of distinct d-operations defined on a set Q of order n is called orthogonal if the system*

$$\{A_i(x_1^d) = a_i\}_{i=1}^k$$

has exactly n^{d-k} solutions for each $a_1^k \in Q^k$.

For $k = 1$ we say that a d-operation A is itself orthogonal . Such d-operation of order n is called *complete* (for this operation the equation $A(x_1^d) = a$ has exactly n^{d-1} solutions for any $a \in Q$, that is the corresponding hypercube has type 0).

Definition 2 ([3]). *A set $\Sigma = \{A_1, A_2, \dots, A_t\}$ of d-operations is called k-wise orthogonal, $1 \le k \le d$, $k \le t$, if every k-tuple $< A_{i_1}, A_{i_2}, \dots, A_{i_k} >$ of distinct d-operations of Σ is orthogonal.*

Each set of complete d-operations is 1-wise orthogonal.

Theorem 1 ([3]). *If a set $\Sigma = \{A_1, A_2, \dots, A_t\}$, $t \ge k$, of d-operations of order n defined on a set Q is k-wise orthogonal with $1 \le k \le d$, then the set Σ is l-wise orthogonal for any l with $1 \le l \le< k$.*

Theorem 2 ([3]). *A d-operation A has type j with $0 \le j \le d-1$ if and only if the set $\Sigma = \{A, E_1^d\}$ is $(j+1)$-wise orthogonal.*

Corollary 1 ([3]). *A d-operation of type j with $0 \le j \le d-1$ has type j_1 for all j_1, $0 \le j_1 < j$.*

In connection with this statement we can consider the maximal type $j_{max}(A) \le d-1$ of a d-operation A (of a corresponding d-hypercube). Using Theorem 2 we conclude that for a d-operation A, $j_{max}(A)$ is the largest j from $0, 1, ..., d-1$ such that the set $\{A, E_1^d\}$ is $(j+1)$-wise orthogonal. By Corollary 1 of [3] $j_{max}(A) = d-1$ for a d-operation A if and only if A is a d-quasigroup.

3. Orthogonal sets of d-ary polynomial operations

Consider more detail orthogonality of a special kind of d-operations, namely, orthogonality of polynomial d-operations of the form

$$A(x_1^d) = a_1x_1 + a_2x_2 + ... + a_dx_d$$

over a field $GF(q)$ (such polynomials are called multilinear).

Let a set $\Sigma = \{A_1, A_2, \ldots, A_t\}$, $d \geq 2$, $t \geq d$, be a set of d-operations each of which is polynomial d-operations over a fields $GF(q)$, that is

$$\begin{aligned} A_1(x_1^d) &= a_{11}x_1 + a_{12}x_2 + ... + a_{1d}x_d, \\ A_2(x_1^d) &= a_{21}x_1 + a_{22}x_2 + ... + a_{2d}x_d, \\ &... \\ A_t(x_1^d) &= a_{t1}x_1 + a_{t2}x_2 + ... + a_{td}x_d. \end{aligned} \tag{1}$$

And let A be the determinant of order $t \times d$, defined by these d-operations.

It is easy to see from Definition 2 that the following statement is valid, where a k-minor is the determinant of $(k \times k)$-sub-array of a determinant A.

Proposition 1. *A set $\Sigma = \{A_1^t\}$, $d \geq 2$, $t \geq d$, of polynomial d-operations of (1) is d-wise orthogonal if and only if all d-minors of the determinant A, defined by these d-operations are different from 0.*

For construction of d-wise orthogonal sets of polynomial d-operations over a field we can use a Vandermonde determinant of order $q-1$ with elements of a field $GF(q)$ [6]. A Vandermonde determinant of order n, $2 \leq n \leq q-1$, is defined in the following way:

$$\Delta_n(a_1, a_2, ..., a_n) = \begin{vmatrix} 1 & a_1 & a_1^2 & ... & a_1^{n-1} \\ 1 & a_2 & a_2^2 & ... & a_2^{n-1} \\ . & . & . & ... & . \\ 1 & a_n & a_n^2 & ... & a_n^{n-1} \end{vmatrix} = \prod_{n \geq i > j \geq 2} (a_i - a_j).$$

Such determinant is not equal 0 if $a_i \neq a_j$, $i \neq j$, and $a_i \neq 0$ for each $i \in \overline{1,n}$. The determinant $\Delta_{q-1}(a_1, a_2, ..., a_{q-1})$ in this case defines an orthogonal $(q-1)$-tuple of polynomial $(q-1)$-operations.

In particular, if a is a primitive element (that is a generating element of multiplicative group of a field), then the determinant

$$\Delta_{q-1}(1, a, a^2, ..., a^{q-2}) = \begin{vmatrix} 1 & 1 & 1 & ... & 1 \\ 1 & a & a^2 & ... & a^{q-2} \\ 1 & a^2 & a^4 & ... & a^{2(n-2)} \\ . & . & . & ... & . \\ 1 & a^{q-2} & a^{2(q-2)} & ... & a^{(q-2)(q-2)} \end{vmatrix}$$

is not equal 0 and defines an $(q-1)$-tuple of polynomial $(q-1)$-operations.

From the considered $(q-1)$-tuples of $(q-1)$-operations we can obtain sets $\Sigma = \{A_1^{q-1}\}$ of d-operations for each d, $2 \leq d < q-1$, if to take $q-1$ of the d-operations corresponding to the first d columns of the determinant $\Delta_{q-1}(a_1, a_2, ..., a_{q-1})$ or $\Delta_{q-1}(1, a, a^2, ..., a^{q-2})$. These sets of d-operations will be d-wise orthogonal by Proposition 4, since all $(d\times d)$-minors are also Vandermonde determinants different from 0.

For an illustration, consider the field $GF(5)$ with elements 0,1,2,3,4, then the $q-1)$=4-tuple of $(q-1)$=4-ary polynomial operations over $GF(5)$ corresponding to the Vandermonde determinant $\Delta_4(1,2,3,4)$ will be the following:

$$\begin{aligned} A_1(x_1^4) &= x_1 + x_2 + x_3 + x_4, \\ A_2(x_1^d) &= x_1 + 2x_2 + 4x_3 + 3x_4, \\ A_3(x_1^d) &= x_1 + 3x_2 + 4x_3 + 2x_4, \\ A_4(x_1^d) &= x_1 + 4x_2 + x_3 + 4x_4. \end{aligned}$$

This 4-tuple defines the 3-wise orthogonal set $\Sigma_1 = \{B_1^4\}$ of ternary operations with $B_1(x_1^3) = x_1+x_2+x_3$, $B_2(x_1^3) = x_1+2x_2+4x_3$, $B_3(x_1^3) = x_1+3x_2+4x_3$, $B_4(x_1^3) = x_1+4x_2+x_3$ and the 2-wise orthogonal set $\Sigma_2 = \{C_1^4\}$ of binary operations where $C_1(x_1^2) = x_1 + x_2$, $C_2(x_1^2) = x_1 + 2x_2$, $C_3(x_1^2) = x_1 + 3x_2$, $C_4(x_1^2) = x_1 + 4x_2$.

Now we give one useful sufficient condition for k-wise orthogonality of a set of polynomial d-operations.

Proposition 2. *Let $\Sigma = \{A_1^t\}$, be a set of polynomial d-operations over a field $GF(q)$, k be a fixed number, $2 \leq k \leq d$ $k \leq t$. The set Σ is k-wise orthogonal if in the determinant of order $k \times d$, defined by each k-tuple of d-operations of Σ there exists at least one k-minor different from 0.*

Proof. Let A be the determinant corresponding to the d-operations of Σ and $< A_{i_1}, A_{i_2}, \ldots, A_{i_k} >$ be a k-tuple of distinct d-operations from Σ. Let in k rows of A corresponding to this k-tuple there exists a k-minor $\overline{A}$ (for simplicity let its k columns are the first ones) which is not equal 0:

$$\overline{A} = \begin{vmatrix} a_{i_1 1} & a_{i_1 2} & \ldots & a_{i_1 k} \\ a_{i_2 1} & a_{i_2 2} & \ldots & a_{i_2 k} \\ . & . & . & \ldots \\ a_{i_k 1} & a_{i_k 2} & \ldots & a_{i_k k} \end{vmatrix} \neq 0$$

.

Then the system of k equations

$$\begin{array}{c} a_{i_1 1}x_1 + a_{i_1 2}x_2 + ...a_{i_1 k}x_k = a_1 - a_{i_1,k+1}x_{k+1} - ... - a_{i_1 d}x_d, \\ a_{i_2 1}x_1 + a_{i_2 2}x_2 + ...a_{i_2 k}x_k = a_2 - a_{i_2,k+1}x_{k+1} - ... - a_{i_2 d}x_d, \\ \dots \\ a_{i_k 1}x_1 + a_{i_k 2}x_2 + ...a_{i_k k}x_k = a_k - a_{i_k,k+1}x_{k+1} - ... - a_{i_k d}x_d \end{array}$$

has exactly one solution for all $a_1, a_2, ..., a_k \in GF(q)$ and for each of q^{d-k}, $(d-k)$-tuples of values of the variables x_{k+1}^d. This means that the system

$$\{A_{i_1}(x_1^d) = a_1, A_{i_2}(x_1^d) = a_2, ..., A_{i_k}(x_1^d) = a_k\}$$

has exactly q^{d-k} solutions. The set Σ is k-wise orthogonal since $i_1, i_2, ..., i_k$ by the condition are arbitrary distinct elements of $\overline{1,t}$. □

Corollary 2. *If a set $\Sigma = \{A_1^t\}$ of polynomial d-operations satisfies the condition of Proposition 2, then a set $\overline{\Sigma} = \{B_1^t\}$ of polynomial s-operations, $s > d$, where*

$$B_i(x_1^s) = A_i(x_1^d) + a_{i,d+1}x_{d+1} + ... + a_{i,s}x_s, i \in \overline{1,t},$$

with arbitrary $a_{i,d+1}, a_{i,d+2}, ..., a_{i,s} \in GF(q)$ is also k-wise orthogonal set.

Proof. In this case the same k-minors different from 0 of the determinant A, defined by Σ, can be used, then the corresponding system of k equations with $s-k$ variables on the right side has a unique solution for q^{s-k} values of these variables. It means that the set $\overline{\Sigma}$ of s-ary operations is k-wise orthogonal. □

Example 1. Consider the set $\Sigma = \{A_1^4\}$ with the following polynomial 4-ary operations over a field $GF(p)$ of a prime order $p \geq 7$:

$$\begin{array}{c} A_1(x_1^4) = x_1 + 2x_2 + 3x_3 + 4x_4, \\ A_2(x_1^d) = 2x_1 + 3x_2 + 4x_3 + 4x_4, \\ A_3(x_1^d) = x_1 + 3x_2 + 6x_3 + 3x_4, \\ A_4(x_1^d) = x_1 + x_2 + x_3 + 5x_4. \end{array}$$

This set of t=4, 4-operations is 3-wise orthogonal. Indeed, it easy to check that in every three rows of the determinant defined by these operations there exists 3-minor different from 0 by $p \geq 7$. Namely, in the triples $< 1, 2, 3 >$, $< 1, 3, 4 >$, $< 2, 3, 4 >$ of rows these 3-minors include the first three columns, and in the triple $< 1, 2, 4 >$ it is 3-minor including the first, the third and the fourth columns. Thus, by Proposition 2 the set Σ is 3-wise orthogonal for any $p \geq 7$.

From this set of four polynomial 4-operations over a field of a prime order $p \geq 7$ by according to Corollary 2 a 3-wise orthogonal set of four polynomial s-operations over the same field can be constructed for $s > 4$.

4. Strongly orthogonal sets of d-ary operations

In [1] it was introduced the notion of a strongly orthogonal set of d-operations. Using Definition 2 we can reformulate this notion of [1] in the following way.

Definition 3. *A set $\Sigma = \{A_1^t\}$, $t \geq 1$, of d-ary operations, given on a set Q, is called strongly orthogonal if the set $\overline{\Sigma} = \{A_1^t, E_1^d\}$ is d-wise orthogonal.*

Note that in the case of a strongly orthogonal set $\Sigma = \{A_1^t\}$ of d-ary operations the number t of d-operations in Σ can be smaller than arity d.

By Theorem 2 each d-operation A_i, $i = 1, 2, ..., t$, of a strongly orthogonal set $\Sigma = \{A_1^t\}$ is a d-quasigroup, has type $j_{max}(A_i) = d - 1$ and any type j_1, $0 \leq j_1 < d - 1$, by Corollary 1. Moreover, a d-operation A is a d-quasigroup if and only if the set $\Sigma = \{A\}$ is strongly orthogonal. A set of d-quasigroups by $d > 2$, $t \geq d$ can be d-wise orthogonal but not strongly orthogonal in contrast to the binary case (d=2).

By Theorem 1 for a strongly orthogonal set Σ of d-operations the set $\overline{\Sigma} = \{A_1^t, E_1^d\}$ is k-wise orthogonal for any k, $1 \leq k \leq d$.

Now we generalize the notion of Definition 3 in the following way.

Definition 4. *Let k be a fixed number, $1 \leq k \leq d$. A set $\Sigma = \{A_1^t\}$, $t \geq 1$, of d-operations is called k-wise strongly orthogonal if the set $\overline{\Sigma} = \{A_1^t, E_1^d\}$ is k-wise orthogonal.*

By $k = d$ we have Definition 3. From the definition of a k-wise strongly orthogonal set and Theorem 2 it follows

Corollary 3. *Let $j_{max}(A)$ be the maximal type of a d-operation A. Then $k - 1 \leq j_{max}(A_i) \leq d - 1$ for each d-operation A_i of a k-wise strongly orthogonal set $\Sigma = \{A_1^t\}$. For every d-operation A_i of a 2-wise strongly orthogonal set $1 \leq j_{max}(A_i) \leq d - 1$.*

From Theorem 1 it immediately follows

Proposition 3. *A k-wise strongly orthogonal set of d-operations is l-wise strongly orthogonal for each l, $1 \leq l < k$.*

Let $< A_1, A_2, ..., A_k >$ be a k-tuple of distinct d-operations. By $< B_1, B_2, ..., B_k >_{\bar{a}}$ we denote the k-tuple of $(d - j)$-retracts, defined by a $2j$-tuple $\bar{a} \in I_j$, of the d-operations $A_1, A_2, ..., A_k$ respectively.

Lemma 1. *Let k be a fixed number, $1 \leq k \leq d$, j be a fixed number, $0 \leq j \leq k-1$, $\{i_1, i_2, ..., i_j\}$ be a fixed subset of $\overline{1,d}$. A k-tuple*

$$T =< A_1, A_2, ..., A_{k-j}, E_{i_1}, E_{i_2}, ..., E_{i_j} >$$

of distinct d-operations, defined on a set Q, is orthogonal if and only if the $(k-j)$-tuple $< B_1, B_2, ..., B_{k-j} >_{\bar{a}}$ of the $(d-j)$-retracts of $A_1, A_2, ..., A_{k-j}$ respectively defined by a tuple $\bar{a} = (i_1, i_2, ..., i_j; a_{i_1}, a_{i_2}, ..., a_{i_j})$ is orthogonal for each of $| Q |^j$ tuples $\bar{a} \in I_j$ with the subset $\{i_1, i_2, ..., i_j\} \subseteq \overline{1,d}$.

Proof. At first we note, that if $k > 1, j = 0$, then $T =< A_1, A_2, ..., A_k >$. By $k = 1$ we have $j = 0$ and orthogonality of the 1-tuple $< A_1 >$ means that the d-operation A_1 is complete. When $j = k-1$ we have a k-tuple $T =< A_1, E_{i_1}, E_{i_2}, ..., E_{i_{k-1}} >$ and orthogonality of T means that the $(d-k+1)$-retract of A_1 is complete.

Let T be an orthogonal k-tuple of d-operations of order n, then by Definition 1 the system

$$\{A_1(x_1^d) = a_1, A_2(x_1^d) = a_2, \ldots, A_{k-j}(x_1^d) = a_{k-j},$$
$$E_{i_1}(x_1^d) = a_{i_1}, E_{i_2}(x_1^d) = a_{i_2}, \ldots, E_{i_j}(x_1^d) = a_{i_j}\} \quad (2)$$

has n^{d-k} solutions for all $a_1, a_2, ...a_{k-j}, a_{i_1}, a_{i_2}, ..., a_{i_j} \in Q$. From this system it follows that

$$x_{i_1} = a_{i_1}, x_{i_2} = a_{i_2}, ..., x_{i_j} = a_{i_j}$$

by the definition of the selectors. Substituting these values in A_i, $i = 1, 2, ..., k-j$, we obtain the $(d-j)$-retracts $B_1, B_2, ..., B_{k-j}$ of $A_1, A_2, ..., A_{k-j}$ respectively defined by the tuple $\bar{a} = (i_1, i_2, ..., i_j; a_{i_1}, a_{i_2}, ..., a_{i_j}) \in I_j$. The $(k-j)$-tuple
$< B_1, B_2, ..., B_{k-j} >_{\bar{a}}$ is orthogonal since the system

$$\{B_1(y_1^{d-j}) = a_1, B_2(y_1^{d-j}) = a_2, \ldots, B_{k-j}(y_1^{d-j}) = a_{k-j}\}$$

has $n^{d-k} = n^{(d-j)-(k-j)}$ solutions for all $a_1, a_2, ..., a_{k-j}$ (since the k-tuple T is orthogonal). It is true for all $(a_{i_1}, a_{i_2}, ..., a_{i_j}) \in Q^j$ by the fixed $\{i_1, i_2, ..., i_j\} \subseteq \overline{1,d}$.

Converse, let each $(k-j)$-tuple $< B_1, B_2, ..., B_{k-j} >_{\bar{a}}$ of $(d-j)$-retracts of d-operations $A_1, A_2, ..., A_{k-j}$, defined by a tuple

$$\bar{a} = (i_1, i_2, ..., i_j; a_{i_1}, a_{i_2}, ..., a_{i_j}) \in I_j$$

with a fixed subset $\{i_1, i_2, ..., i_j\} \subseteq \overline{1,d}$ for some elements $a_{i_1}, a_{i_2}, ..., a_{i_j} \in Q$ is orthogonal. This means that the system

$$\{B_1(y_1^{d-j}) = a_1, B_2(y_1^{d-j}) = a_2, \ldots, B_{k-j}(y_1^{d-j}) = a_{k-j}\}$$

has $n^{(d-j)-(k-j)} = n^{d-k}$ solutions for all $a_1, a_2, ..., a_{k-j} \in Q$ and the system (2) has n^{d-k} solutions for all $a_1, a_2, ... a_{k-j} \in Q$ and the fixed $a_{i_1}, a_{i_2}, ..., a_{i_j} \in Q$. The same we have fixing any another j-tuple $(a'_{i_1}, a'_{i_2}, ..., a'_{i_j}) \in Q^j$ and obtaining another $(k-j)$-tuple of $(d-j)$-retracts defined by the tuple $\bar{a}' = (i_1, i_2, ..., i_j; a'_{i_1}, a'_{i_2}, ..., a'_{i_j}) \in I_j$. Thus, the k-tuple T is orthogonal.

□

Let k (j) be a fixed number, $1 \leq k \leq d$ $(0 \leq j \leq k-1)$. Denote by $\Sigma_{\bar{a}} = \{B_1, B_2, ..., B_t\}$ the set of the $(d-j)$-retracts of d-operations from a set $\Sigma = \{A_1, A_2, ..., A_t\}$, defined by a fixed tuple

$$\bar{a} = (i_1, i_2, ..., i_j; a_{i_1}, a_{i_2}, ..., a_{i_j}) \in I_j.$$

Theorem 3. *Let k be a fixed number, $1 \leq k \leq d$. A set $\Sigma = \{A_1^t\}$ of d-operations, defined on a set Q, is k-wise strongly orthogonal if and only if for each j, $0 \leq j \leq k-1$, if $t \geq k$ (for each j, $k-t \leq j \leq k-1$, if $t < k$) and for each $\bar{a} \in I_j$ the set $\Sigma_{\bar{a}} = \{B_1, B_2, ..., B_t\}$ of the $(d-j)$-retracts of $A_1, A_2, ..., A_t$, defined by $\bar{a}$, is $(k-j)$-wise orthogonal.*

Proof. Let a set $\Sigma = \{A_1^t\}$ be k-wise strongly orthogonal, that is the set $\overline{\Sigma} = \{A_1^t, E_1^d\}$ is k-wise orthogonal by Definition 3. It means that each k-tuple

$$< A_{l_1}, A_{l_2}, ..., A_{l_{k-j}}, E_{i_1}, E_{i_2}, ..., E_{i_j} >$$

is orthogonal for each j, $0 \leq j \leq k-1$, if $t \geq k$ (for each j, $k-t \leq j \leq k-1$, if $t < k$) and for each subset $\{l_1, l_2, ..., l_{k-j}\} \subseteq \overline{1,t}$. By Lemma 1 it follows that the $(k-j)$-tuple $< B_{l_1}, B_{l_2}, ..., B_{l_{k-j}} >_{\bar{a}}$ of the $(d-j)$-retracts of $A_{l_1}, A_{l_2}, ..., A_{l_{k-j}}$ is orthogonal for each $\bar{a} \in I_j$ and for each $\{l_1, l_2, ..., l_{k-j}\} \subseteq \overline{1,t}$. It means that the set $\Sigma_{\bar{a}}$ is $(k-j)$-wise orthogonal for each $\bar{a} \in I_j$ and for each j, $0 \leq j \leq k-1$ if $t \geq k$ (for each j, $k-t \leq j \leq k-1$, if $t < k$).

Converse, let each set $\Sigma_{\bar{a}}$ of $(d-j)$-retracts of the d-operations from Σ is $(k-j)$-wise orthogonal for each j, $0 \leq j \leq k-1$, if $t \geq k$ (for each j, $k-t \leq j \leq k-1$, if $t < k$) and each $\bar{a} \in I_j$. Then each k-tuple

$$< A_{l_1}, A_{l_2}, ..., A_{l_{k-j}}, E_{i_1}, E_{i_2}, ..., E_{i_j} >$$

is orthogonal by Lemma 1 for any suitable j and any $l_1, l_2, ..., l_{k-j} \subseteq \overline{1,t}$. It means that the set $\overline{\Sigma} = \{A_1^t, E_1^d\}$ is k-wise orthogonal and the set Σ is k-wise strongly orthogonal. □

For a d-wise strongly orthogonal set according to Theorem 3 by $k = d$ and Theorem 1 we have

Corollary 4. *If a set $\Sigma = \{A_1^t\}$ of d-operations is d-wise strongly orthogonal, then the set $\Sigma_{\bar{a}} = \{B_1^t\}$ of the $(d-j)$-retracts of $A_1, A_2, ..., A_t$ is $(d-j)$-wise orthogonal (and j_1-wise orthogonal for each j_1, $1 \leq j_1 \leq d-j$) for each j, $0 \leq j \leq d-1$, if $t \geq d$ (for each j, $d-t \leq j \leq d-1$, if $t < d$) and for each $\bar{a} \in I_j$.*

As it was said above, all d-operations of a strongly orthogonal set are d-quasigroups, so we shall consider only sets of polynomial d-quasigroups (in this case all mappings $x_j \to a_{ij}x_j$ are permutations) by establishment of criterion for strongly orthogonality of a set of polynomial operations.

Proposition 4. *A set $\Sigma = \{A_1^t\}$ of polynomial d-quasigroups, $d \geq 2$, with the determinant A over a field is strongly orthogonal if and only if all k-minors for each k, $2 \leq k \leq d$, if $t \leq d$ (for each k, $2 \leq k \leq t$, if $t < d$) of A is not equal 0.*

Proof. By Definition 3 and Theorem 3 a set Σ is strongly orthogonal if and only if for each j, $0 \leq j \leq d-1$, if $t \geq d$ (for each j, $d-t \leq j \leq d-1$, if $t < d$) the set $\Sigma_{\bar{a}} = \{B_1, B_2, ..., B_t\}$ of the $(d-j)$-retracts of $A_1, A_2, ..., A_t$, defined by $\bar{a} \in I_j$ is $(d-j)$-wise orthogonal. By Proposition 1 this holds by $d-j \geq 2$ if and only if all $(d-j)$-minors of the determinant A are not equal 0. For $j = d-1$ $(d-j = 1)$ we have the set $\Sigma_{\bar{a}}$ of 1-ary operations which are permutations in the case of d-quasigroups, so composes an 1-wise orthogonal set. □

Example 2. Let $(Q, +, \cdot)$ be the field of a prime order $p = 17$ or $p > 19$. Consider the polynomial ternary quasigroups

$$A_1(x_1^3) = 2x_1 + 2x_2 + 3x_3,$$

$$A_2(x_1^3) = 5x_1 + 4x_2 + 3x_3,$$

$$A_3(x_1^3) = x_1 + 6x_2 + 5x_3.$$

By Proposition 4 the set $\Sigma = \{A_1, A_2, A_3\}$ is strongly orthogonal, since it is easy to check that the 3-minor and all 2-minors of the respective determinant are different from 0.

Now we consider k-wise strongly orthogonal sets of polynomial d-operations. At first we remind that from Theorem 3 it follows that each $(d-k+1)$-retract of each d-operation of k-wise strongly orthogonal set is complete. Taking this into account, we shall consider only such d-operations by establishment the following sufficient condition for k-wise strongly orthogonal set of polynomial d-operations.

Proposition 5. *Let k be a fixed number, $2 \leq k \leq d$, $\Sigma = \{A_1^t\}$ be a set of polynomial d-operations over a field with the determinant A. The set Σ is k-wise strongly orthogonal if*

(i) all $(d-k+1)$-retracts of each d-operations of Σ are complete;

(ii) for each j, $0 \leq j \leq k-2$, if $t \geq k$ (for each j, $k-t \leq j \leq k-2$, if
$t < k$) in every $k-j$ rows of the determinant A without any j columns there exists a $(k-j)$-minor different from 0.

Proof. Let $\bar{a} = (i_1, i_2, ..., i_j; a_{i_1}, a_{i_2}, ..., a_{i_j}) \in I_j$ and $\Sigma_{\bar{a}} = \{B_1, B_2, ..., B_t\}$ be the set of the $(d-j)$-retracts of $A_1, A_2, ..., A_t$, then the set $\Sigma_{\bar{a}}$ corresponds to the determinant $\overline{A}$ of order $t \times (d-j)$ which is the determinant A without fixed j columns $i_1, i_2, ..., i_j$ (by any $a_{i_1}, a_{i_2}, ..., a_{i_j}$, since the corresponding system must be solved for any right parts of the equations). If in each $k-j$ rows of the determinant $\overline{A}$ there exists at least one $(k-j)$-minor different from 0, then by Proposition 2 the set $\Sigma_{\bar{a}}$ is $(k-j)$-wise orthogonal for j, $0 \leq j \leq k-2$. If $j = k-1$, $\Sigma_{\bar{a}}$ consists of $(d-k+1)$-retracts which by (i) are complete and so 1-wise orthogonal. Thus , by Theorem 3 Σ is k-wise strongly orthogonal. □

Example 3. We shall illustrate Proposition 13 at the set $\Sigma = \{A_1, A_2, A_3\}$ of the following three polynomial 4-ary operations (quasigroups):

$$A_1(x_1^4) = x_1 + 2x_2 + 3x_3 + 4x_4,$$

$$A_2(x_1^4) = 2x_1 + 3x_2 + 4x_3 + 4x_4,$$

$$A_3(x_1^4) = x_1 + 3x_2 + 6x_3 + 3x_4$$

over the field $GF(p)$ of a prime order $p \geq 7$. Check by Proposition 5 that the set Σ is 3-wise strongly orthogonal.

In this case $d = 4$, $k = t = 3$, $0 \leq j \leq 1$. All $(d-k+1) = 2$-retracts of every 4-operation of Σ are complete since these operations are 4-quasigroups.

If $j = 0$, then $k - j = 3$ and the 3-minor in the determinant A defined by Σ with the first three columns is different from 0.

If $j = 1$, then $k - j = 2$. In this case it is easy to check that in A without any one of four columns, in each two rows there exists a 2-minor different from 0.

Thus, be Proposition 5 the set Σ is 3-wise strongly orthogonal.

5. Uniformly orthogonal sets of d-ary operations

Two d-hypercubes, $d \geq 2$, H_1 and H_2 is called j-uniformly orthogonal if when superimposed and any j, $0 \leq j \leq d-2$, coordinates are fixed, the resulting subarrays of dimention $d-j$ are themselves orthogonal. This notion of the j-uniformly orthogonality of two d-hypercubes naturally leads to the following concept for d-operations, if we take into account that an fixation of coordinates in a hypercube H leads to a retract of the corresponding operation A_H.

Definition 5 . *Two d-operations A_1 and A_2 of order n is called j-uniformly orthogonal for fixed j, $0 \leq j \leq d-2$, if the pair $(B_1, B_2)_{\bar{a}}$ of the $(d-j)$-retracts of operations A_1,A_2 respectively, defined a tuple $\bar{a} = (i_1, i_2, ..., i_j; a_{i_1}, a_{i_2}, ..., a_{i_j}) \in I_j$ is orthogonal (that is, by the definition, the system $\{B_l(y_1^{d-j}) = a_1, B_2(y_1^{d-j}) = a_2\}$ has $n^{(d-j)-2}$ solutions for all $a_1, a_2 \in Q$ and for each tuple $\bar{a} \in I_j$).*

Definition 5. *A set $\Sigma = \{A_1^t\}$, $t \geq 2$, of d-operations is called (2-wise) j-uniformly orthogonal, $0 \leq j \leq d-2$, if any two operations of Σ are j-uniformly orthogonal.*

Proposition 6. *A set $\Sigma = \{A_1^t\}$ of d-operations is (2-wise) j-uniformly orthogonal if and only if the $(2+j)$-tuple $< A_{l_1}, A_{l_2}, E_{i_1}, E_{i_2}, ..., E_{i_j} >$ is orthogonal for each subset $\{i_1, i_2, ..., i_j\} \subseteq \overline{1,d}$ and for all $l_1, l_2 \in \overline{1,t}$, $l_1 \neq l_2$.*

Proof. This follows from Definitions 5 and 6 and Lemma 1. □

Now we generalize the notion of Definitions 5 and 6 in the following way.

Definition 6. *Let k be a fixed number, $1 \leq k \leq d$, and j be a fixed number, $0 \leq j \leq d-k$. A k-tuple $< A_1, A_2, ..., A_k >$ of distinct d-operations is called j-uniformly orthogonal if the k-tuple $< B_1, B_2, ..., B_k >_{\bar{a}}$ of the $(d-j)$-retracts of $A_1, A_2, ..., A_k$, defined by a tuple*

$$\bar{a} = (i_1, i_2, ..., i_j; a_{i_1}, a_{i_2}, ..., a_{i_j}) \in I_j,$$

is orthogonal for each $\bar{a} \in I_j$.

Definition 7. *Let k, j be fixed numbers, $1 \leq k \leq d$, $0 \leq j \leq d-k$. A set $\Sigma = \{A_1^t\}$, $t \geq k$, of d-operations is called k-wise j-uniformly orthogonal if each k-tuple of distinct d-operations from Σ is j-uniformly orthogonal (the same, if the set $\Sigma_{\overline{a}}$ of the $(d-j)$-retracts of d-operations from Σ is k-wise orthogonal for any $\overline{a} \in I_j$).*

It is easy to see that 0-uniformly orthogonality of a k-tuple $< A_1^k >$ means that this k-tuple is itself orthogonal ($I_0 = \emptyset$) and a k-wise 0-uniformly orthogonal set is simply k-wise orthogonal.

If $k = d$, then j=0 and a set Σ is d-wise orthogonal.

In the case $j = d - k$ we have

$$I_{d-k} = \{(i_1, i_2, ..., i_{d-k}; a_{i_1}, a_{i_2}, ..., a_{i_{d-k}})\}$$

and all k-tuples of $(d - (d - k)) = k$- retracts

$$< B_1(y_1^k), B_2(y_1^k), \ldots, B_k(y_1^k) >_{\bar{a}}$$

of $A_1, A_2, ..., A_k$ are orthogonal , when $\bar{a} \in I_{d-k}$. Taking this into account, we obtain that if $\Sigma = \{A_1^t\}$, $t \geq k$, of d-operations is a k-wise $(d - k)$-uniformly orthogonal set, then the set $\Sigma_{\bar{a}} = \{B_1, B_2, ..., B_t\}$ of the k-retracts of $A_1, A_2, ..., A_t$, defined by $\bar{a}$, is k-wise orthogonal for each $\bar{a} \in I_{d-k}$.

By k=1 we obtain an 1-wise j-uniformly orthogonal set $\Sigma = \{A_1^t\}$, $t \geq 1$, of d-operations, it means that every operation A_i of Σ has type j and $j \leq j_{max}(A_i) \leq d - 1$ (see Theorem 2).

Proposition 7. *Let k, j be fixed numbers, $1 \leq k \leq d$, $0 \leq j \leq d - k$. A set $\Sigma = \{A_1^t\}$, $t \geq k$, of d-operations is k-wise j-uniformly orthogonal if and only if the $(k + j)$-tuple $(1 \leq k + j \leq d)$*

$$< A_{s_1}, A_{s_2}, ..., A_{s_k}, E_{i_1}, E_{i_2}, ..., E_{i_j} >$$

is orthogonal for all $\{s_1, s_2, ..., s_k\} \subseteq \overline{1,t}$ and for all $\{i_1, i_2, ..., i_j\} \subseteq \overline{1,d}$.

Proof. Let a set Σ be k-wise j-uniformly orthogonal. Then by Definitions 7 and 8 each k-tuple $< B_{s_1}, B_{s_2}, ..., B_{s_k} >_{\bar{a}}$ of the operations $A_{s_1}, A_{s_2}, ..., A_{s_k}$ from Σ, defined by a tuple $\bar{a} = (i_1, i_2, ..., i_j; a_{i_1}, a_{i_2}, ..., a_{i_j}) \in I_j$, is orthogonal for each subset $\{i_1, i_2, ..., i_j\} \subseteq \overline{1,d}$ and for each tuple $(a_{i_1}, a_{i_2}, ..., a_{i_j}) \in Q^j$. Now use Lemma 1.

Converse, if a $(k + j)$-tuple $< A_{s_1}, A_{s_2}, ..., A_{s_k}, E_{i_1}, E_{i_2}, ..., E_{i_j} >$ is orthogonal for all subsets $S = \{s_1, s_2, ..., s_k\} \subseteq \overline{1,t}$ and for all $I = \{i_1, i_2, ..., i_j\} \subseteq \overline{1,d}$, then by Lemma 1 each $(k + j - j) = k$-tuple $< B_{s_1}, B_{s_2}, ..., B_{s_k} >_{\bar{a}}$ of the $(d - j)$-retracts of $A_{s_1}, A_{s_2}, ..., A_{s_k}$ is orthogonal for all subsets S of $\overline{1,t}$, for all subsets I of $\overline{1,d}$ and all $\bar{a} \in I_j$. Thus, the set Σ is k-wise j-uniformly orthogonal by Definitions 7 and 8. □

Corollary 5. *Each k-wise j-uniformly orthogonal set is l-wise j_1-uniformly orthogonal for each l, $1 \leq l \leq k$, and for each j_1, $0 \leq j_1 \leq j$.*

Proof. From Theorem 1 it follows that each $(l+j_1)$-tuple

$$< A_{s_1}, A_{s_2}, ..., A_{s_l}, E_{i_1}, E_{i_2}, ..., E_{i_{j_1}} >$$

is orthogonal for all l, $1 \leq l \leq k$, for all j_1, $0 \leq j_1 \leq j$, for all $\{s_1, s_2, ..., s_l\} \subseteq \overline{1,t}$ and for all $\{i_1, i_2, ..., i_{j_1}\} \subseteq \overline{1,d}$. Now use Proposition 7 for the $(l+j_1)$-tuples. □

Corollary 6. *Let $j_{max}(A)$ denote the maximal type of a d-operation A, $1 \leq k \leq d$, $0 \leq j \leq d-k$, $\Sigma = \{A_1^t\}$ be a k-wise j-uniformly orthogonal set of d-operations. Then*

$$j \leq j_{max}(A_i) \leq d-1$$

for each d-operation A_i of Σ.

Proof. From Proposition 7 and Corollary 5 it follows that $(1+j)$-tuple $< A_{s_1}, E_{i_1}, E_{i_2}, ..., E_{i_j} >$ is orthogonal for each d-operation $A_{s_1} \in \Sigma$ and each
$\{i_1, i_2, ..., i_j\} \subseteq \overline{1,d}$. Thus, the set $\{A_{s_1}, E_1^d\}$ is $(j+1)$-wise orthogonal and by Theorem 2 the operation A_{s_1} has at least type j. □

Corollary 7. *For each d-operation A_i of an 1-wise $(d-1)$-uniformly orthogonal set $\Sigma = \{A_1^t\}$, $j_{max}(A_i) = d-1$, that is A_i is a d-quasigroup.*

Proof. In this case $k=1$, $j=d-1$ and $j_{max}(A_i) = d-1$ by Corollary 6. But by Corollary 1 of [3] a d-operation has type $j = d-1$ if and only if it is a d-quasigroup. □

For a set of polynomial d-operations over a field the following sufficient condition of k-wise j-uniformly orthogonality can be given.

Proposition 8. *Let $\Sigma = \{A_1^t\}$, $d \geq 2$, be a set of polynomial d-operations over a field $GF(q)$ with the determinant A, k, j be an fixed number, $2 \leq k \leq d$, $0 \leq j \leq d-k$ $k \leq t$. Then Σ is k-wise j-uniformly orthogonal if in each k rows of A without any j columns there exists k-minor different from 0.*

Proof. According to Definition 8 the set Σ is k-wise j-uniformly orthogonal if and only if the set $\Sigma_{\overline{a}}$ of $(d-j)$-retracts of the d-operations from Σ is k-wise orthogonal by any $\overline{a} \in I_j$. Now use Proposition 2 for the set $\Sigma_{\overline{a}}$, which corresponds to the determinant A without j columns. □

Example 4. Using this proposition we give an example of 3-wise 1-uniformly orthogonal set 5-ary operations over a field $GF(q)$ with a prime $q \geq 7$. Let $\Sigma = \{A_1, A_2, A_3, A_4\}$, where

$$A_1(x_1^5) = x_1 + x_2 + x_3 + x_4 + x_5,$$
$$A_2(x_1^5) = 2x_1 + 3x_2 + 5x_3 + 4x_4 + x_5,$$
$$A_3(x_1^4) = 3x_1 + 2x_2 + 4x_3 + x_4 + 2x_5,$$
$$A_4(x_1^4) = x_1 + 4x_2 + 3x_3 + 2x_4 + 3x_5.$$

In this case $d = 5$, $t = 4$,$j = 1$. It is easy to check that by fixation the columns with numbers 1,2,4 and 5 in the corresponding determinant A of Σ the 3-minors in any three rows with the first three possible columns is not equal 0. By fixation the column with number 3 in rows 1,2,3 the 3-minor in columns 1,2,5 is not equal 0, whereas for rows 1,3,4 and 2,3,4 the 3-minors in columns 1,2,4 are not equal 0.

The following theorem establishes a connection between k-wise strongly orthogonal and l-wise j-uniformly orthogonal sets.

Theorem 4. *Let k be a fixed number, $1 \leq k \leq d$. A k-wise strongly orthogonal set of d-operations is l-wise j-uniformly orthogonal for each l, $1 \leq l \leq k$, and for each j, $0 \leq j \leq k - l$.*

Proof. Let a set $\Sigma = \{A_1^t\}$, $k \leq t$, be k-wise strongly orthogonal. Then by Definition 4 the set $\overline{\Sigma} = \{A_1^t, E_1^d\}$ is k-wise orthogonal, so each k-tuple

$$< A_{s_1}, A_{s_2}, ..., A_{s_l}, E_{i_1}, E_{i_2}, ..., E_{i_{k-l}} >$$

is orthogonal for all l, $1 \leq l \leq k$, for each subset $\{s_1, s_2, ..., s_l\} \subseteq \overline{1,t}$ and for each subset $\{i_1, i_2, ..., i_{k-l}\} \subseteq \overline{1,d}$. By Proposition 7 the set Σ is l-wise $(k-l)$-uniformly orthogonal and by Corollary 5 is l-wise j-uniformly orthogonal for each j, $0 \leq j < k - l$. □

Thus, from Theorem 4 it follows that a k-wise strongly orthogonal set Σ is
1-wise $0-, 1-, ...$ and $(k-1)$-uniformly orthogonal,
2-wise $0-, 1-, ...$ and $(k-2)$-uniformly orthogonal,
3-wise $0-, 1-, ...$ and $(k-3)$-uniformly orthogonal,...,
$(k-2)$-wise 0-,1- and 2-uniformly orthogonal,
$(k-1)$-wise 0- and 1-uniformly orthogonal,
k-wise 0-uniformly orthogonal.

So, for the 3-wise strongly orthogonal set $\Sigma = \{A_1, A_2, A_3\}$ of the 4-ary operations in Example 3 we have that Σ is
1-wise 0-,1- and 2-uniformly orthogonal,
2-wise 0- and 1-uniformly orthogonal,
3-wise 0-uniformly orthogonal.

From Theorem 4 by $k = d$ immediately it follows

Corollary 8. *A strongly orthogonal set of d-operations is l-wise j-uniformly orthogonal for each l, $1 \leq l \leq d$, and for each j, $0 \leq j \leq d - l$.*

So, in Example 2 the strongly orthogonal set $\Sigma = \{A_1, A_2, A_3\}$ of ternary operations is
1-wise 0-,1- and 2-uniformly orthogonal,
2-wise 0- and 1-uniformly orthogonal,
3-wise 0-uniformly orthogonal.

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