Algebra and Discrete Mathematics RESEARCH ARTICLE
Number 3. **(2005).** pp. 56 – 59


# On square-Hamiltonian graphs

**K. D. Protasova**

Communicated by V. M. Usenko

Abstract. A finite connected graph $G$ is called square-Hamiltonian if $G^2$ is Hamiltonian. We prove that any join of the family of Hamiltonian graphs by tree is square-Hamiltonian. Applying this statement we show that the line graph and any round-about reconstruction of an arbitrary finite connected graph is square-Hamiltonian.

Let $G$ be a finite connected graph with the set of vertices $V(G)$ and the set of edges $E(G)$. Given any vertices $u, v \in V(G)$, we denote by $d(u, v)$ the length of the shortest path between $u$ and $v$. A graph $G$ is called *Hamiltonian* if there exists the numeration $v_1, \dots, v_n$ of $V(G)$ such that

$$d(v_1, v_2) = ... = d(v_{n-1}, v_n) = d(v_n, v_1) = 1.$$

By [3], the cube $G^3$ of every finite connected graph is Hamiltonian. In other words, there exists the numeration $v_1, \dots, v_n$ of $V(G)$ such that

$$d(v_1, v_2) \leq 3, \dots, d(v_{n-1}, v_n) \leq 3, \; d(v_n, v_1) \leq 3.$$

For application of this fact to the partitions of groups see [5, Chapter 3].

A finite connected graph $G$ is called *square-Hamiltonian* if $G^2$ is Hamiltonian, i.e. there exists the numeration $v_1, \dots, v_n$ of $V(G)$ such that

$$d(v_1, v_2) \leq 2, \dots, d(v_{n-1}, v_n) \leq 2, \; d(v_n, v_1) \leq 2.$$

---

**2000 Mathematics Subject Classification:** *05C45.*

**Key words and phrases:** *square-Hamiltonian graphs, join of graphs, line graph, round-about reconstruction.*

By Fleishner's Theorem, every finite 2-connected graph is square-Hamiltonian. A graph $G$ is called *2-connected* if the graph

$$G[V(G) \setminus \{a\}]$$

is connected for every $v \in V(G)$, where $G[A]$ is the induced subgraph of $G$ with the set of vertices $A \subseteq V(G)$. This theorem was proved in the chain of papers ended in [2]. The proof of Fleishner's Theorem was simplified by Riha [7]. This proof can be find also in [1, Theorem 10.3.1]. Another sufficient condition was done by Mathews and Sumner [4]: every $K_{1,3}$-free graph is square-Hamiltonian. By [6], a finite tree is square-Hamiltonian if and only if there exists the path $v_1, \dots, v_k$ in $G$ such that $G[V(G) \setminus \{v_1, \dots, v_k\}]$ is the disjoint union of singletons. For proof see also [5, Theorem 4.1].

In this paper we consider some general construction, the join of the family of graphs by tree, which allows us to produce the square-Hamiltonian graphs from the Hamiltonian graphs. Applying this construction we show that the line graph and any round-about reconstruction of an arbitrary finite connected graph is square-Hamiltonian. These statements are not covered by the above sufficient conditions for graph to be square-Hamiltonian.

Let $G_1, \dots, G_n$ be connected graphs, $T$ be a finite tree, $V(T) = \{v_1, \dots, v_n\}$. It is supposed that the sets $V(G_1), \dots, V(G_n)$, $V(T)$ are pairwise disjoint and $|V(G_i)| \geq \rho(v_i)$, $i \in \{1, ..., n\}$, where $\rho(v)$ is the degree of $v$. For every edge $(v_i, v_j) \in E(T)$, we take some vertices $u \in U_i$, $u' \in U_j$ and introduce the new edge $(u, u')$. It must be done in such a way that any two new edges corresponding to distinct edges of tree have no common vertices. The resulting connected graph $G$ is called the join of $G_1, \dots, G_n$ by $T$. Clearly, $G_1, \dots, G_n$ are the induced subgraphs of $G$. It should be mentioned that we can obtain some distinct joins from the fixed family $G_1, \dots, G_n$ and $T$.

**Theorem.** *A join $G$ of any family of Hamiltonian graphs $G_1, \dots, G_n$ by an arbitrary tree $T$ is square-Hamiltonian.*

*Proof.* Every vertex $v \in V(G)$ is the vertex of some graph $G_i$. We say that $v$ is isolated if either $|V(G_i)| = 1$ or $v$ is adjacent in $G$ to only vertices from $V(G_i)$. We show that $G$ has at least one isolated vertex. To this end we take an arbitrary terminal vertex $v_k$ of $T$. If $|V(G_k)| = 1$, $V(G_k) = \{v\}$, then $v$ is isolated. If $|V(G_k)| > 1$, we choose the vertex $u$, which is adjacent to some vertex from $V(G_j)$, $j \neq k$. Then all vertices from $V(G_k) \setminus \{u\}$ are isolated.

Using induction by $n$, we prove that, for every isolated vertex $v$ of $G$, there exist the adjacent vertex $u$ and the Hamiltonian circle in $G^2$

with the first vertex $v$ and the last vertex $u$. For $n = 1$ the statement is evident because $G_1$ is Hamiltonian. Let we have proved the statement for all joins by trees with $< n$ vertices.

Hamiltonian circle in $G^2$ with the first vertex $v$ and the last vertex $u$. For $n = 1$ the statement is evident because $G_1$ is Hamiltonian. Let we have proved the statement for all joins by trees with less than $n$ vertices.

Let $V(T) = \{v_1, \ldots, v_n\}$ and $v$ is an isolated vertex of $G$. Then $v$ is the vertex of some graph $G_i$ and we consider two cases.

Case $|V(G_i)| = 1$. Since $v$ is isolated, $v_i$ is a terminal vertex of $T$. Let $u$ be the vertex of $G$ adjacent to $v_i$. After deletion of $v$ we get the graph $\Gamma$, which is the join by tree with $n-1$ vertices, and $u$ is isolated in $\Gamma$. By the inductive hypothesis, there exists the Hamiltonian circle $u_1, \ldots, u_m$ in $\Gamma^2$ such that $u = u_1$ and $(u_1, u_m) \in E(\Gamma)$. Then $v, u_m, u_{m-1}, ..., u_1$ is the Hamiltonian circle in $G^2$.

Case $|V(G_i)| > 1$. Let $u_1, \ldots, u_k$ be the Hamiltonian circle in $G_i$ such that $u_1 = v$. After deletion of the edges $E(G_i)$ we get the pairwise disjoint (by vertices) graphs $\Gamma_1, \ldots, \Gamma_k$ such that $u_1 \in \Gamma_1, \ldots, u_k \in \Gamma_k$, $V(\Gamma_1) = \{u_1\}$. Every graph $\Gamma_j$ is the union by tree with $\leq n$ vertices and $u_j$ is terminal vertex of $\Gamma_j$. By Case 1, for every $j \in \{2, ..., k\}$, there exists a Hamiltonian circle in $\Gamma_j^2$ with the first vertex $u_j'$ adjacent to $u_j$ and the last vertex $u_j$. Then $u_1, C_2, \ldots, C_n$ is the Hamiltonian circle in $G^2$. □

Let $G$ be a graph. For the set of vertices of the *line graph* $L(G)$ we take $E(G)$ and $(e_1, e_2) \in E(L(G))$ if and only if the edges $e_1, e_2$ have the common vertex in $G$.

**Corollary 1.** *The line-graph $L(G)$ of every finite connected graph is square Hamiltonian.*

*Proof.* First we suppose that $G$ is a tree. For every vertex $v \in V(G)$, let $K(v)$ be the complete graph with $\rho(v)$ vertices. Then $L(G)$ is the join of the family $\{K(v) : v \in V(G)\}$ of Hamiltonian graphs by the tree $T$ and we can apply Theorem. If $G$ has a circle $C$, we take two adjacent vertices $u, v$ of $G$, delete the edge $(u, v)$, but add new edges $(u, w)$, $w \notin V(G)$. After this reconstruction the set of vertices of $L(G)$ does not change but the set of edges decreases, so we can reduce the general case to the case of trees. □

For every finite connected graph $G$, we describe its round-about reconstruction. First, we replace every edges $(u, v) \in E(G)$ by three edges $(u, u')$, $(u, v')$, $(v', v)$, where the new vertices $(u', v')$, do not belong to $V$. Second, we delete every vertex $u \in V$ but connect in some circle all new vertices adjacent to $u$. The resulting graph $R(G)$ is called a *round-about reconstruction* of $G$. It should be mentioned that the fixed graph $G$ could have a few round-about reconstructions, because on the second step we have some possibilities to organize the circle.

**Corollary 2.** *Every round-about reconstruction $R(G)$ of an arbitrary finite connected graph $G$ is square-Hamiltonian.*

*Proof.* If $G$ is a tree, then $R(G)$ is the join of the family of circles by the tree $G$, and we can apply Theorem. If $G$ has a circle $C$, we take two adjacent vertices $u, v$ from $C$, replace the edge $(u, v)$ by the edges $(u, u')$, $(v, v')$, where the new vertices $u', v'$ are terminal in the obtained graph $G'$. If a round-about reconstruction of $G'$ is square-Hamiltonian, then $R(G)$ is square-Hamiltonian. Thus, we can reduce the general case to the case of trees. □

## References

[1] R. Diestel, Craph Theory, Graduated Text in Math., Springer-Verlag, 2000.

[2] H. Fleishner, The square of every 2-connected graph is Hamiltonian, J.Comb. Theory B, 16(1974), 29-34.

[3] J. Karganis, On the cube of graph, Canad. Math.Bull., 11(1969), 295-296.

[4] M. Mathews, D. Sumner, Hamiltonian results in $K_{1,3}$ - free graphs, J. Graph Theory, 8 (1984), 139-146.

[5] I.Protasov, T.Banakh, Ball Structures and Colorings of Graphs and Groups, Mat. Stud. Monogr. Ser., 11, VNTL, Lviv, 2003.

[6] К.Д. Протасова, Квазігамільтонові графи, Вісник Київського ун-ту, серія фіз.-мат. науки, 2003, N 1, 45-50.

[7] S. Riha, A new proof of the theorem of Fleishner, J.Comb. Theory B, 52 (1991), 117-123.

Contact information

**K. D. Protasova** Department of Cybernetics, Kyiv University, Volodimirska 64, Kyiv 01033, Ukraine
*E-Mail:* kseniya@profit.net.ua

Received by the editors: 15.08.2005
and final form in 10.09.2005.