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Diagonalizability theorems for matrices over rings with finite stable range

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Abstract. We construct the theory of diagonalizability for matrices over Bezout ring with finite stable range. It is shown that every commutative Bezout ring with compact minimal prime spectrum is Hermite. It is also shown that a principal ideal domain with stable range 1 is Euclidean domain, and every semilocal principal ideal domain is Euclidean domain. It is proved that every matrix over an elementary divisor ring can be reduced to "almost" diagonal matrix by elementary transformations.

Introduction

The aim of this note is to study the problem of diagonalizability for matrices over rings with finite stable range. The theorem that each matrix over a ring R is equivalent to some diagonal matrix was proved in the case when R is the ring of integers in 1861 by Henry J.Stephen Smith [1]. It was consequently extended by Dickson [2], Wedderburn [3], van der Waerden [4] and Jacobson [5] for various commutative and noncommutative Euclidean domains and commutative PIDs (principal ideal domains) and later for noncommutative PIDs by O.Teichmuller [6] (then in somewhat sharper form in Asano [7] and Jacobson [8]). The theorem is also known for an arbitrary PIR (principal ideal rings)[9].

Key words and phrases: *finite stable range, elementary divisor ring, Hermite ring, ring with elementary reduction of matrices, Bezout ring, minimal prime spectrum.*

Following Kaplansky[10] a ring R is said to be an elementary divisor ring if every matrix over R is equivalent to a diagonal matrix. Kaplansky proved that if R is an elementary divisor ring then every finitely presented R-module is a direct sum of cyclic modules. In [11] the converse to Kaplansky's theorem for commutative ring is proved.

By [10] a ring is said to be Hermite if every 1 by 2 and 2 by 1 matrix over it is equivalent to a diagonal matrix. Obviously, an elementary divisor ring is Hermite and it is easy to see that an Hermite ring is Bezout (a ring is a Bezout ring if every finitely generated 1-sided ideal is principal). Examples that neither implication is reversible are provided by Gillman and Henriksen in [12]. In [10] Kaplansky proved that a Bezout ring is Hermite when all zero divisors of the ring are in the Jacobson radical establishing in particular the fact that all Bezout domains are Hermite [13]. Henriksen [14] changed Kaplansky's hypothesis to the assumption that the Jacobson radical contains a prime ideal and proved that the theorem is still valid.

Adequate domains were introduced by Helmer in [15]. It had been known that principal ideal domains were elementary divisor rings and Helmer showed that the less restrictive hypothesis that an integral domain be adequate is sufficient. Kaplansky [10] began the consideration of adequate rings with zero divisors by showing that the adequate ring whose zero divisors are in the Jacobson radical is an elementary divisor ring. M.Henriksen [16] proves that if R is a unit regular ring then every matrix over R admits diagonal reduction. The diagonalizability question for rectangular matrices was answered by Menal and Moncasi [17], who showed that all rectangular matrices over given regular ring R admits a diagonal reduction if and only if R is Hermite. If R is a separative regular ring, then every square matrix over R admit a diagonal reduction [18]. In [19,20] it is proved that any left distributive ring R is an elementary divisor ring if and only if R is invariant. Menal and Moncasi [17] showed that any right Hermite and left Bezout ring is left Hermite. Further, the stable range (in the sense of K-theory) of a right or left Hermite ring is at most 2 [17].

In the present paper we construct the theory of diagonalizability for matrices over Bezout ring with finite stable range.

In [14] Henriksen asked whether every commutative semilocal Bezout ring is Hermite. This question was answered affirmatively in [11]. In the situation of noncommutative ring this question was answered affirmatively in [21].

Henriksen has raised the following question: If R is a commutative Bezout ring with compact minimal prime spectrum, is R Hermite. Despite of results of [22], we show that every commutative Bezout ring with compact minimal prime spectrum is Hermite.

Following [23] a ring is said to be a right n-Hermite if every 1 by n matrix over it is equivalent to a diagonal matrix. We show that a right n-Hermite ring has stable range $\leq n$. And if R is a right Bezout ring with finite stable range n then R is a right $n+1$-Hermite ring. We show if R is left n-Hermite and right Bezout then R is right n-Hermite.

In [24] the problem of investigation of rings with elementary reduction of matrices is posed. A ring is said to be a ring with elementary reduction of matrices if every matrix over R can be reduced to diagonal form by using only elementary transformations. Clearly, every ring with elementary reduction of matrices is an elementary divisor ring. But there exists an elementary divisor ring which is not a ring with elementary reduction of matrices (e.g., the ring $\mathbf{R}[x,y]/(x^2+y^2+1)$ where $\mathbf{R}$ is the ring of reals)[24,25,35]. Obviously, every Euclidean domain is a ring with elementary reduction of matrices. We show that over an elementary divisor ring, every matrix can be reduced to "almost" diagonal matrix by elementary transformations.

1. Definitions

Furtheron R will always denote a ring (associative, but not necessary commutative) with $1 \neq 0$. We shall write R_n for the ring of n by n matrices with elements in R. By a unit of a ring we mean an element with two-sided inverse. Units of R_n will be said to be unimodular. If $b = ca$ we say that a is a right divisor of b; equivalent conditions are $b \in Ra$ and $Rb \subseteq Ra$. We say that a is a total divisor of b if $RbR \subseteq Ra \cap aR$, or in words: everything in the two-sided ideal generated by b is right and left divisible by a. It is observed that an element is not necessary a total divisor of itself. If R is commutative then right, left and total divisibility all coincide.

An n by m matrix $A = (a_{ij})$ is said to be diagonal if $a_{ij} = 0$ for all $i \neq j$. We say that a matrix A admits diagonal reduction if there exist the unimodular matrices P, Q such that PAQ is a diagonal matrix. We shall call two matrices A and B over a ring R equivalent(notation $A \sim B$) if there exist the unimodular matrices P, Q such that $B = PAQ$. If every matrix over R is equivalent to a diagonal matrix (d_{ij}) with the property that every d_{ii} is a total divisor of d_{i+1i+1} then R is an elementary divisor ring. We recall that R is said to be right(left) Hermite if every 1 by 2 (2 by 1) matrix admits diagonal reduction, and if both, R is an Hermite ring. If every 1 by n (n by 1) matrix admits diagonal reduction then R is a right (left) n-Hermite ring.

A row $(a_1, \ldots, a_n)$ over a ring R is called right unimodular if a_1R+

$\dots + a_nR = R$. If $(a_1, \dots a_n)$ is a right unimodular n-row over a ring R then we say that $(a_1, \dots, a_n)$ is reducible if there exists an $(n-1)$-row $(b_1, \dots, b_{n-1})$ such that the $(n-1)$-row $(a_1 + a_nb_1, \dots a_{n-1} + a_nb_{n-1})$ is right unimodular. A ring R is said to have stable range n if n is the least positive integer such that every right unimodular $(n+1)$-row is reducible. This number is denoted by $s.r.(R)$.

By a right (left) Bezout ring we mean a ring in which all finitely generated right (left) ideals are principal, and by a Bezout ring a ring which is both right and left Bezout.

A commutative ring R is said to be adequate if R is Hermite and for $a, b \in R$ with $a \neq 0$ there exist $r, s \in R$ such that $a = rs, rR + bR = R$ and if a nonunit s' divides s then $s'R + bR \neq R$. Under a right k-stage division chain for elements $a, b \in R, b \neq 0$ we understand a sequence of equalities:

$$a = bq_1 + r_1, b = r_1q_2 + r_2, \dots, r_{k-2} = r_{k-1}q_k + r_k.$$

A finite right division chain is by the definition a right k-stage division chain for some $k \in \mathbf{N}$. A norm over domain R is a function $N : R \longrightarrow \mathbf{Z}$ such that $N(0) = 0$ and $N(a) > 0$ for each $a \neq 0$. A domain R is called a right n-Euclidean domain with respect to a norm N if for any elements $a, b \in R, b \neq 0$ there exists a right k-stage division chain such $N(r_k) < N(b)$. Obviously, any right 1-Euclidean domain is a right Euclidean domain. A domain R is called a right ω-Euclidean domain with respect to a norm N if for any elements $a, b \in R, b \neq 0$ there are $k \in \mathbf{N}$ and a right k-stage division chain such that $N(r_k) < N(b)$. Any right ω-Euclidean domain is a right Bezout domain. R is said to be regular if for every $a \in R$ there exists an $x \in R$ such that $axa = a$. A regular ring is said to be unit regular if for any $a \in R$ there exists a unit $u \in R$ such that $aua = a$.

We denote by $GL_n(R)$ the group of units of R_n. We write $GE_n(R)$ for the subgroup of $GL_n(R)$ generated by elementary matrices. The Jacobson radical of a ring R will be denoted by $J(R)$. Denote by $U(R)$ the group of units of R.

2. The space of minimal prime ideals

Let R be a commutative ring. By a minimal prime ideal of R we shall mean a proper prime ideal that contains no smaller prime ideal. Thus, for example, if R is an integer domain than (0) is the only minimal prime ideal of R. Let $\min(R)$ be the minimal prime spectrum of R. If $x \in R$, define $D(x) = \{P \in \min R \mid x \notin P\}$. Then the sets of the form $D(x)$

form a basis for the Zariski topology of $\min R$. When we say that R is compact we mean that it is compact in this topology. Concerning rings with compact minimal spectrum see [26-30].

In order to obtain a characterization of commutative Bezout ring with compact minimal spectrum we need some preliminary results.

Theorem 1. *[21, Theorem 1]. A commutative Bezout ring R is an Hermite ring if and only if R has stable range ≤ 2.*

We denote by N the nilradical of R.

Proposition 1. *If R is a commutative Bezout ring, then*

$$s.r.(R) = s.r.(R/N).$$

Proof. By [31] $s.r.(R/N) \leq s.r.(R)$. Set $\overline{R} = R/N$ and let $s.r.(\overline{R}) \leq n$. Let $a_1R + \cdots + a_nR + a_{n+1}R = R$ then $\overline{a_1}\,\overline{R} + \cdots + \overline{a_n}\,\overline{R} + \overline{a_{n+1}}\,\overline{R} = \overline{R}$. Since $s.r.(\overline{R}) \leq n$, we obtain $(\overline{a_1} + \overline{a_{n+1}x_1})\overline{R} + \cdots + (\overline{a_n} + \overline{a_{n+1}x_n})\overline{R} = \overline{R}$. Say $(\overline{a_1} + \overline{a_{n+1}x_1})\overline{v_1} + \cdots + (\overline{a_n} + \overline{a_{n+1}x_n})\overline{v_n} = \overline{1}$ for some $\overline{v_1}, \ldots, \overline{v_n} \in \overline{R}$. We get $(a_1 + a_{n+1}x_1)v_1 + \cdots + (a_n + a_{n+1}x_n)v_n = 1 + m$ for some $x_1, \ldots x_n, v_1, \ldots v_n \in R$ and $m \in N$. Obviously, $1 + m \in U(R)$. Then $s.r.(R) \leq n$. $\square$

Theorem 2. *Let R be a commutative Bezout ring with compact minimal prime spectrum then R is Hermite.*

Proof. We begin by showing that R/N is semihereditary. Every localization of R/N is a semiprime valuation ring that is a valuation domain, and it follows that every ideal of R/N is flat. Thus it will suffice by [32] to prove that the classical quotient ring R/N is regular. But this follows from [30, proposition 9] because every finitely generated faithful ideal of R/N contains a non-zero divisor. Since every semihereditary Bezout ring is Hermite [11, Theorem 2.4] then $s.r.(R/N) \leq 2$. By proposition 1 then $s.r.(R) \leq 2$, i.e. R is a commutative Bezout ring with stable range ≤ 2. By theorem 1 R is a Hermite ring. $\square$

3. A right n-Hermite ring

Proposition 2. *If R is a right(left) n-Hermite ring then R has stable range n.*

Proof. Let $a_1R + \cdots + a_nR + a_{n+1}R = R$. Since R is a right n-Hermite ring, $(a_1, \ldots, a_n)P = (d, 0, \ldots, 0)$ for some $d \in R, P = (p_{ij}) \in GL_n(R)$. Let $P^{-1} = (\alpha_{ij}) \in GL_n(R)$. We claim that $(a_1 + a_{n+1}\alpha_{n1}) + \cdots + (a_n +$

$a_{n+1}\alpha_{nn})$ is a right unimodular row. We have $(a_1 + a_{n+1}\alpha_{n1})p_{1n} + \cdots + (a_n + a_{n+1}\alpha_{nn})p_{nn} = a_1p_{1n} + \cdots + a_np_{nn} + a_{n+1}(\alpha_{n1}p_{1n} + \cdots + \alpha_{nn}p_{nn}) = 0 + a_{n+1} \cdot 1 = a_{n+1}$ and $(a_1 + a_{n+1}\alpha_{n1})p_{11} + \cdots + (a_n + a_{n+1}\alpha_{nn})p_{n1} = a_1p_{11} + \cdots + a_np_{n1} + a_{n+1}(\alpha_{n1}p_{11} + \cdots + \alpha_{nn}p_{n1}) = d + a_{n+1} \cdot 0 = d$. and $a_{n+1}, d \in (a_1 + a_{n+1}\alpha_{n1})R + \cdots + (a_n + a_{n+1}\alpha_{nn})R$. Since $(a_1, \ldots, a_n)P = (d, 0, \ldots, 0)$, we obtain $a_1R + \cdots + a_nR = dR$. On the other hand, we have $a_1R + \cdots + a_nR + a_{n+1}R = R$ then $dR + a_{n+1}R = R$. Since $a_{n+1}, d \in (a_1 + a_{n+1}\alpha_{n1})R + \cdots + (a_n + a_{n+1}\alpha_{nn})R$, we see that $(a_1 + a_{n+1}\alpha_{n1})R + \cdots + (a_n + a_{n+1}\alpha_{nn})R = R$ and $s.r.(R) = n$. Because the stable range of a ring coincides with the stable range of its opposite ring [31, Theorem 2] the result also follows if R is left n-Hermite. $\square$

Proposition 3. *If R is left n-Hermite and right Bezout ring then R is right n-Hermite.*

Proof. Let $Ra_1 + \cdots + Ra_n = R$, then $P\begin{pmatrix} a_1 \\ \vdots \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ for some $P \in GL_n(R)$. Clearly, $\begin{pmatrix} a_1 \\ \vdots \\ \vdots \\ a_n \end{pmatrix} = P^{-1}\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Let $P^{-1} = (p_{ij})$, then $a_1 = p_{11}, \ldots, a_n = p_{n1}$. Now we obtain that every left unimodular n-column is the first column of unimodular n by n matrix over R. We first prove the analogous result for rows. If $a_1R + \cdots + a_nR = R$ then $a_1u_1 + \cdots + a_nu_n = 1$ for some $u_1, \ldots u_n \in R$ and there exists an unimodular n by n matrix Q of the form $Q = \begin{pmatrix} u_1 & \\ \vdots & * \\ u_n & \end{pmatrix}$. Clearly, $(a_1, \ldots, a_n)Q = (1, b_2, \ldots, b_n)$. If

$$P = \begin{pmatrix} 1 & -b_2 & -b_3 & \cdots & -b_n \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

then $P \in GE_n(R)$ and $(a_1, \ldots, a_n)QP = (1, 0, \ldots, 0)$. The row $(a_1, \ldots, a_n)$ is the first row of the matrix $P^{-1}Q^{-1} \in GL_n(R)$. Now we prove that R is right n-Hermite. Let $a_1, \ldots, a_n \in R$ then, since R is right Bezout

$a_1R+\cdots+a_nR=dR$, say $a_i=da_i^0,(i=1,\ldots,n), d=a_1u_1+\cdots+a_nu_n$. We get $d(a_1^0+\cdots+a_n^0u_n-1)=0$, so for some $c\in R$ such that $dc=0$ we have $a_1^0R+\cdots+a_n^0R+cR=R$. It follows from proposition 2 that R has stable range $\le n$ thus $(a_1^0+cv_1)R+\cdots+(a_n^0+cv_n)R=R$ for some $v_1,\ldots v_n\in R$. By the above we can find a unimodular matrix Q of the form

$$Q=\begin{pmatrix} a_1^0+cv_1 & \cdots & a_n^0+cv_n \\ & * & \end{pmatrix}.$$

Clearly, $(a_1,\ldots,a_n)=(d,0,\ldots,0)Q$ then $(a_1,\ldots,a_n)Q^{-1}=(d,0,\ldots,0)$ so R is right n-Hermite. □

Proposition 4. *If R is a right n-Hermite ring then R is right Bezout.*

Proof. Let $a,b\in R$, then since R is a right n-Hermite, $(a,b,0,\ldots,0)P=(d,0,\ldots,0)$ for some $d\in R$ where $P=(p_{ij})\in GL_n(R)$. We get $ap_{11}+bp_{21}=d$.

Let $P^{-1}=(\alpha_{ij})\in GL_n(R)$. Clearly, $(a,b,\ldots,0)=(d,0,\ldots,0)P^{-1}$, then $a=d\alpha_{11}, b=d\alpha_{21}$ and $aR\subset dR, bR\subset dR$. Then $aR+bR\subset dR$ and $aR+bR=dR$. Therefore, R is right Bezout. □

Corollary 1. *Let R be a right Bezout ring with finite stable range n, then for any $a_1,\ldots,a_m\in R$ where $m\ge n+1$ there exists a unimodular matrix $P\in GE_m(R)$ such that $(a_1,\ldots,a_m)P=(d,0,\ldots,0)$ for some $d\in R$.*

Proof. We first prove that any right unimodular row of length m over R can be completed to a unimodular matrix. If $a_1R+\cdots+a_mR=R$, then there exists $m-1$-row $(v_1,\ldots,v_{m-1})$ such that $(a_1+a_mv_1)R+\cdots+(a_{m-1}+a_mv_{m-1})R=R$. There exist $u_1,\ldots,u_{m-1}\in R$ such that $(a_1+a_mv_1)u_1+\cdots+(a_{m-1}+a_mv_{m-1})u_{m-1}=1$.

Set

$$P_1=\begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_1 & v_2 & \ldots & v_{m-1} & 1 \end{pmatrix}\in GE_n(R);$$

$$P_2=\begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & u_1(1-a_m) \\ 0 & 1 & \ldots & 0 & u_2(1-a_m) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & u_{m-1}(1-a_m) \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix}\in GE_{n+1}(R).$$

We see that for a row $(a_1,\ldots,a_m)P_1P_2$ there exists a matrix $P_3\in GE_m(R)$ such that $(a_1,\ldots,a_m)P_1P_2P_3=(1,0,\ldots,0)$. Thus we obtain a matrix $P \in GE_m(R)$ such that $(a_1, \ldots, a_m)P = (1, 0, \ldots, 0)$. Then $(a_1, \ldots, a_m)$ is the first row of the matrix P^{-1}. Since R is a right Bezout ring, for any $a_1, \ldots, a_m \in R$ there exists $d \in R$ such that $a_1R + \cdots + a_mR = dR$. Say $a_1u_1 + \cdots + a_mu_m = d, a_1 = da_1^0, \ldots, a_m = da_m^0$. From these relations we get $d(a_1^0u_1 + \cdots + a_m^0u_m - 1) = 0$, so that $a_1^0R + \cdots + a_m^0R + cR = R$ for some $c \in R$ such that $dc = 0$. Since $s.r.(R) < m$, we see that $(a_1^0 + cv_1)R + \cdots + (a_m^0 + cv_m)R = R$, where $v_1, \ldots, v_m \in R$. By the above we can find a unimodular matrix $P \in GE_m(R)$ of the form

$$P = \begin{pmatrix} a_1^0 + cv_1 & \cdots & a_m^0 + cv_m \\ & * & \end{pmatrix}.$$

Clearly, $(a_1, \ldots, a_m)P^{-1} = (d, 0, \ldots, 0)$. $\square$

4. Bass' first stable range condition

Bass' lowest (the first) stable range condition [34] asserts the following: if a and b in R satisfy $Ra + Rb = R$ then there exists t in R with $a + tb$ left invertible. More exactly this is the "left" version of the condition, and there is a symmetric "right" version, but the two versions are in the fact equivalent [34].

We shall now discuss the question of the uniqueness of the generators of principal right ideals. If $a = bu$, where u is a unit, we say that a and b are right associates. Clearly, associate elements are right multiples of each other or they generate the same principal right ideals. We raise the converse question: If $aR = bR$, are a and b necessarily right associate? It is well known that the answer is affirmative if there are no divisors of 0 [8]. Kaplansky [10] extended this result to the ring in which all right divisors of 0 are in the radical. We shall prove this for ring with stable range 1.

Proposition 5. *Let R be a ring with stable range 1. Then $aR = bR$ implies that a, b are right associate.*

Proof. We have $a = by, b = ax$ so $a = axy$. If $a = b = 0$ there is nothing to prove. Otherwise $a(1 - xy) = 0$. Let $1 - xy = c$, then $xR + cR = R, ac = 0$. Since $s.r.(R) = 1$, we have $x + cv = u \in U(R)$ for $v \in R$. Thus $ax + acv = au$. Then $ax - au = b$ and $bu^{-1} = a$. $\square$

Corollary 2. *Let R be a ring of stable range 1. If $A_1, A_2 \in R_n$ are matrices which are right multiples of each other. Then A_1, A_2 are right associate.*

Proof. Since for any natural number n $s.r.(R) = 1$ if and only if so is R_n [34, Theorem 2.4], by theorem 5 the proof of corollary is obvious. $\square$

Proposition 6. *Let R be a right Bezout ring of stable range 1. Then for any $a, b \in R$ there exist $x \in R, d \in R$ such that $a + bx = d$ and $aR + bR = dR$.*

Proof. Since R is a right Hermite ring [21, Theorem 2], for any a, b there exist $\delta, a_0, b_0 \in R$ such that $a = \delta a_0, b = \delta b_0$ and $a_0R + b_0R = R$. Since $s.r.(R) = 1$, there exist $x \in R, u \in U(R)$ such that $a_0 + b_0x = u$. Then $a + bx = \delta u$. Obviously, $\delta uR = \delta R$. Set $d = \delta u$, then $a + bx = d$ and $aR + bR = dR$. □

Proposition 7. *Let R be a right Bezout domain with stable range 1. Then R is a right 2-Euclidean domain.*

Proof. Let N be a function $R \longrightarrow \mathbf{Z}$ such that $N(0) = 0$ and $N(a) = 1$ for each $a \neq 0$. Let $a, b \in R, b \neq 0$, by proposition 6, there exist $x \in R, d \in R$ such that $a + bx = d$ and $aR + bR = dR$, then $a = da_0, b = db_0$ for some $a_0, b_0 \in R$. Thus $a = b \cdot (-x) + d, b = db_0 + 0$ and $N(0) < N(b)$. □

Proposition 8. *Let R be a principal ideal domain with stable range 1. Then R is Euclidean domain.*

Proof. Let $|a|$ denote the number of prime factors of $a \in R \setminus 0$ in the factorization of a into prime factors. Obviously, $|a| \geq 0$ and $|ab| = |a| + |b|$. By proposition 6 there exist $x \in R, d \in R$ for $a, b \in R, b \neq 0$ such that $a + bx = d$ and $aR + bR = dR$. Let $b = db_0$ for $b_0 \in R$. If $|d| < |b|$ then $a = b(-x) + d$. If $|d| = |b|$ then $|b_0| = 0$ and $b_0 \in U(R)$. Since $a = da_0$ for some $a_0 \in R$, we see that $a = bb_0^{-1}a_0$, i.e. $aR \subset bR$. □

Since every semilocal ring is a ring with stable range 1, we have

Corollary 3. *Every semilocal principal ideal domain is a Euclidean domain.*

5. A ring with elementary reduction of matrices

Recall that a ring is said to be a ring with elementary reduction of matrices if every matrix can be reduced to a diagonal form by using only elementary transformations [24].

Proposition 9. *Let R be a right Bezout ring and for any elements $a, b \in R$ there exists a unimodular matrix $Q \in GE_2(R)$ such that $Q\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} d \\ 0 \end{pmatrix}$ for $d \in R$. Then there exists a unimodular matrix $P \in GE_2(R)$ such that $(a, b)P = (c, 0)$ for any $a, b \in R$.*

Proof. Let $Ra + Rb = R$. Then there exists unimodular 2 by 2 matrix $Q \in GE_2(R)$ such that $Q\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then we have

$$\begin{pmatrix} a \\ b \end{pmatrix} = Q^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1}$$

where $Q^{-1} \in GE_2(R)$. Let $Q^{-1} = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$. By (1) we have $a = q_{11}, b = q_{21}$. Then $\begin{pmatrix} a \\ b \end{pmatrix}$ is the first column of the matrix $Q^{-1} \in GE_2(R)$ i.e. any left unimodular column of length 2 over R can be completed to a unimodular matrix in $GE_2(R)$. Since R is left Hermite and right Bezout, R is a right Hermite ring [17, Proposition 8].

Let $aR + bR = R$, then there exist $u, v \in R$ such that $au + bv = 1$. We know that the left unimodular column $\begin{pmatrix} u \\ v \end{pmatrix}$ can be completed to a unimodular matrix $U \in GE_2(R)$. Then $(a, b)U = (1, c)$ for a suitable element $c \in R$. We see that for the row $(a, b)U$ there exists a unimodular matrix $V \in GE_2(R)$ such that $(a, b)UV = (1, 0)$. Thus we obtain a unimodular matrix $P \in GE_2(R)$ such that $(a, b)P = (1, 0)$. Then (a, b) is the first row of the matrix $P^{-1} \in GE_2(R)$.

Since R is Hermite, for every pair of elements $a, b \in R$ the following holds: there exist $d, a', b' \in R$ such that $a = da', b = db'$ and $a'R + b'R = R$. By the above argument we can find a unimodular matrix $P \in GE_2(R)$ such that $(a', b')P = (1, 0)$. Clearly, $(a, b)P = (d, 0)$, which finishes the proof. □

Corollary 4. *If R is a right ω-Euclidean Bezout ring then R is a left ω-Euclidean ring.*

Proof. If R is a right ω-Euclidean ring then for any elements $a, b \in R$ there exists a unimodular matrix $Q \in GE_2(R)$ such that $Q\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} d \\ 0 \end{pmatrix}$ [35, Proposition 1]. By proposition 9 there exists a unimodular matrix $P \in GE_2(R)$ such that $(a, b)P = (c, 0)$ for any $a, b \in R$.

By [35, Proposition 4] R is a left ω-Euclidean ring. □

Theorem 3. *Let R be an elementary divisor ring then, for any n by m matrix A $(n > 2, m > 2)$ we can find unimodular matrices $P \in GE_n(R)$*

and $Q \in GE_m(R)$ *such that*

$$PAQ = \begin{pmatrix} \epsilon_1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \epsilon_2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \epsilon_s & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & & & \\ 0 & 0 & \cdots & 0 & & A_0 & \\ 0 & 0 & \cdots & 0 & & & \end{pmatrix},$$

where ϵ_i *is a total divisor of* ϵ_{i+1}, $1 \le i \le s-1$ *and* A_0 *– 2 by k or k by 2 matrix.*

Proof. Since R is an elementary divisor ring, we can find a unimodular matrix P_1 such that

$$P_1 A = \begin{pmatrix} a'_{11} & \cdots & a'_{2m} \\ & * & \end{pmatrix}$$

where $a'_{11}R + \cdots + a'_{1m}R = \epsilon_1 R$, where ϵ_1 is a total divisor of all the elements of P_1A. Let $P_1 = (p'_{ij}) \in GE_n(R)$. Obviously, $p'_{11}R + \cdots + p'_{1n}R = R$. Since any elementary divisor ring is Hermite, $s.r.(R) = 2$. Since $n > 2$ by [33, Proposition 1], the right unimodular row $(p'_{12}, \ldots, p'_{1n})$ can be completed to a unimodular matrix $H_1 \in GE_n(R)$. Then

$$H_1 A = \begin{pmatrix} a'_{11} & \cdots & a'_{1m} \\ * & \cdots & * \end{pmatrix}.$$

Since $m > 2$, there exists a unimodular matrix $S_1 \in GE_m(R)$ such that

$$H_1 A S_1 = \begin{pmatrix} \epsilon_1 & 0 & \cdots & 0 \\ & * & & \end{pmatrix}.$$

We may now use elementary transformations to sweep out the first column of H_1AS_1 and we obtain

$$\begin{pmatrix} \epsilon_1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A_1 & \\ 0 & & & \end{pmatrix},$$

where ϵ_1 is still a total divisor of every elements of A_1. Proceeding in this way we complete the reduction. □

Theorem 4. *Let R be an elementary divisor ring. Then for every n by m matrix A, where $m-n=2$ we can find a unimodular matrices $P \in GL_n(R),\ Q \in GE_m(R)$ such that PAQ is a diagonal matrix*

$$diag(d_1, \dots, d_r, 0, \dots, 0),$$

where d_i is a divisor of $d_{i+1}, i = 1, 2, \dots, r-1$.

Proof. By theorem 3 we need only to consider the case of a 2 by 4 matrix A. Since R is an elementary divisor ring, there exists a unimodular matrix $P \in GL_2(R)$ such that

$$PA = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix},$$

with $a_{11}R + a_{12}R + \ldots + a_{14}R = \epsilon R$, where ϵ is a total divisor of all the elements of PA. Since any elementary divisor ring is Hermite, R is a ring with stable range ≤ 2. By corollary 1 there exist a unimodular matrix $Q \in GE_4(R)$ such that

$$PAQ = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ * & * & * & * \end{pmatrix}.$$

We may now use elementary transformations to sweep out the first column of PAQ and we obtain

$$\begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & b_{22} & b_{23} & b_{24} \end{pmatrix},$$

where ϵ is a total divisor of every elements b_{22}, b_{23}, b_{24}.

By corollary 1, there exists a unimodular matrix $W \in GE_3(R)$ such that $(b_{22}, b_{23}, b_{24})W = (b, 0, 0)$.

Then

$$\begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & b_{22} & b_{23} & b_{24} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & W & \\ 0 & & & \end{pmatrix} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & b & 0 & 0 \end{pmatrix},$$

where, obviously, ϵ is a divisor of b. □

Proposition 10. *Let R be a commutative adequate ring then for every nonsingular n by n matrix A we can find unimodular matrices $P \in GE_n(R), Q \in GL_n(R)$ such that*

$$PAQ = \begin{pmatrix} \epsilon_1 & 0 & \cdots & 0 \\ 0 & \epsilon_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \epsilon_n \end{pmatrix},$$

where ϵ_i *is a divisor of* $\epsilon_{i+1}, 1 \leq i \leq n-1$.

Proof. Let $n = 2$, without loss of generality we may change notations and assume that the greatest common divisor off all elements of A is 1. Since R is Hermite, $s.r.(R) = 2$ and we find a unimodular matrix $Q_1 \in GL_2(R)$ such that $AQ_1 = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$. Since A is a nonsingular, $c \neq 0$. Write $c = rs$ where $rR + aR = R$ and if a nonunit element s' divides s then $s'R + aR \neq R$. Then, obviously, $(ra + b)R + cR = R$. Multiplying the first row of a matrix AQ_1 by r and adding it to the second row, we obtain the matrix

$$A_1 = \begin{pmatrix} a & 0 \\ ra + b & c \end{pmatrix}.$$

Since $(ra+b)R+cR = R$, there exist a unimodular matrix $Q_2 \in GL_2(R)$ such that

$$A_1Q_2 = \begin{pmatrix} * & * \\ 1 & 0 \end{pmatrix}.$$

The matrix A_1Q_2 is reducible by elementary transformations to the form $\begin{pmatrix} 1 & 0 \\ 0 & \Delta \end{pmatrix}$. Application of theorem 3 completes the proof of this proposition. □

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