Algebra and Discrete Mathematics
Number 1. **(2005).** pp. 122 – 132

RESEARCH ARTICLE

Wreath product of Lie algebras and Lie algebras associated with Sylow p-subgroups of finite symmetric groups

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Dedicated to Yu.A. Drozd on the occasion of his 60th birthday

ABSTRACT. We define a wreath product of a Lie algebra L with the one-dimensional Lie algebra L_1 over $\mathbb{F}_p$ and determine some properties of this wreath product. We prove that the Lie algebra associated with the Sylow p-subgroup of finite symmetric group S_{p^m} is isomorphic to the wreath product of m copies of L_1. As a corollary we describe the Lie algebra associated with Sylow p-subgroup of any symmetric group in terms of wreath product of one-dimensional Lie algebras.

1. Introduction

Lie rings associated to a group are already the classical objects of modern algebra. One can find their usefulness in a variety of applications, including the restricted Burnside problem, the study of some group identities, the theory of fixed point of automorphism, the coclass theory for p-groups and pro-p groups, the investigation of just-infinite pro-p groups, and the recent study of Hausdorff dimension and the spectrum of pro-p groups. Lie ring methods provide a recipe for translating some group-theoretic questions to Lie-theoretic ones.

A classical operation in group theory is the wreath product of groups. The wreath product of Lie algebras was defined by A. L. Shmelkin [6]

2000 Mathematics Subject Classification: *17B30, 17B60, 20F18, 20F40.*
Key words and phrases: *Lie algebra, wreath product, semidirect product, Lie algebra associated with the lower central series of the group, Sylow p-subgroup, symmetric group.*

already in 1973. In spite of this the notion is almost non-investigated by now.

We define another notion of a wreath product of a Lie algebra with the one-dimensional Lie algebra over the finite field $\mathbb{F}_p$. Our idea of the construction comes from the study of some class of Lie algebras associated with p-groups, namely the Sylow p-subgroups of finite symmetric groups. The Sylow p-subgroup P_m of the symmetric group S_{p^m} is isomorphic to a wreath product of cyclic groups of order p [7]. The structure of the Lie algebra associated with P_m was investigated in [9]. Our definition of wreath product allows us to prove the main result of the article: Lie algebra associated with the Sylow p-subgroup of finite symmetric group S_{p^m} is isomorphic to the wreath product of one-dimensional Lie algebra, i.e.

$$L(C_p \wr \ldots \wr C_p) = L(C_p) \wr \ldots \wr L(C_p).$$

Using this theorem we describe the Lie algebra associated with the Sylow p-subgroup of any finite symmetric group S_n in terms of wreath product of one-dimensional Lie algebra. Also we investigate some basic properties of our definition of wreath product.

2. The wreath product of Lie algebras and its properties

Recall the definition of the semidirect product of Lie algebras (see [1]).

Let M and N be Lie algebras over K and $a \mapsto \varphi_a$ be a homomorphism from M to the Lie algebra of differentiations of the algebra N. Define a Lie bracket on the direct sum L of K-modules M and N by the equality:

$$([a, b], [a', b']) = [(a, a'), (b, b') + \varphi_a(b') - \varphi_{a'}(b)],$$

where $a, a' \in M$ and $b, b' \in N$.

Definition 1. Lie algebra L is called the semidirect product of algebra M and algebra N which corresponds to the homomorphism $\varphi : M \to \mathcal{D}(N)$, and we denote it as $L = M \leftthreetimes_\varphi N$.

Let L be a Lie Algebra over the field $\mathbb{F}_p$ and L_1 be the one-dimensional Lie algebra over $\mathbb{F}_p$.

Let $L[x]/\langle x^p \rangle$ be the Lie algebra of polynomials over L of degree at most $p-1$. The Lie bracket of the monomials in this algebra is defined in the following way:

$$(lx^n, l'x^m) = \begin{cases} (l, l')x^{n+m}, & \text{if } n+m < p; \\ 0, & \text{if } n+m \geq p. \end{cases} \quad (1)$$

By linearity the Lie bracket is determined for all polynomials.

The following proposition determines the one-to-one correspondence between the set $L[x]/\langle x^p\rangle$ and the set of all maps from L_1 to L.

Proposition 1. Every map $f : L_1 \to L$ corresponds to the unique polynomial $q(x)$ over L of degree at most $p-1$ such that $f(\alpha) = q(\varepsilon(\alpha))$, where $\varepsilon : L_1 \to \mathbb{F}_p$ is the some isomorphism of vector spaces.

Proof. Let $f(\alpha_0)$, …, $f(\alpha_{p-1})$ be the images of the elements of Lie algebra L_1 under the map $f : L_1 \to L$. Consider the linear system of equalities with respect to $l_0, \dots, l_{p-1} \in L$:

$$\begin{array}{rcl} l_{p-1}\varepsilon(\alpha_0)^{p-1} + \ldots + l_1\varepsilon(\alpha_0) + l_0 & = & f(\alpha_0) \\ l_{p-1}\varepsilon(\alpha_1)^{p-1} + \ldots + l_1\varepsilon(\alpha_1) + l_0 & = & f(\alpha_1) \\ \vdots & & \vdots \\ l_{p-1}\varepsilon(\alpha_{p-1})^{p-1} + \ldots + l_1\varepsilon(\alpha_{p-1}) + l_0 & = & f(\alpha_{p-1}), \end{array}$$

where $\{\varepsilon(\alpha_i)\}$ are all elements of the field $\mathbb{F}_p$.

Or we may write down it as:

$$\begin{pmatrix} \varepsilon(\alpha_0)^{p-1} & \ldots & \varepsilon(\alpha_0) & 1 \\ \varepsilon(\alpha_1)^{p-1} & \ldots & \varepsilon(\alpha_1) & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \varepsilon(\alpha_{p-1})^{p-1} & \ldots & \varepsilon(\alpha_{p-1}) & 1 \end{pmatrix} \begin{pmatrix} l_{p-1} \\ \vdots \\ l_0 \end{pmatrix} = \begin{pmatrix} f(\alpha_0) \\ \vdots \\ f(\alpha_{p-1}) \end{pmatrix}$$

Determinant of the matrix $det(A)$ is the Vandermond determinant and thus is nonzero. Hence, there is only one set of elements $l_{p-1}, \dots, l_0$ for arbitrary $f(\alpha_0), \dots f(\alpha_{p-1})$. That is

$$(l_{p-1}, \dots, l_0)^T = A^{-1}(f(\alpha_0), \dots f(\alpha_{p-1}))^T.$$

Thus, to every map $f : L_1 \to L$ corresponds the unique polynomial $q(x) = l_{p-1}x^{p-1} + \ldots + l_1 x + l_0$ over L and by construction $f(\alpha) = q(\varepsilon(\alpha))$. □

Therefore exists the bijection between the set of all maps $f : L_1 \to L$ and the set of all polynomials over L of degree at most $p-1$. The structure of Lie algebra $L[x]/\langle x^p\rangle$ defines the structure of Lie algebra on the set of all maps $f : L_1 \to L$. We will denote this Lie algebra as $Fun(L_1, L) \simeq L[x]/\langle x^p\rangle$.

The identification ε between L_1 and $\mathbb{F}_p$ gives us the structure of L_1-module on the algebra $Fun(L_1, L)$. Thus, we also consider the Lie algebra $Fun(L_1, L)$ as L_1–module.

Further we will not distinguish the notations of the elements of one-dimension Lie algebra L_1 and the field $\mathbb{F}_p$. From the context it is clear from which structures the elements are considered.

Let $f \in Fun(L_1, L)$. Denote by $f' \in Fun(L_1, L)$ the derivative of the polynomial f.

Proposition 2. For every $\alpha \in L_1$ the map $D_\alpha : Fun(L_1, L) \to Fun(L_1, L)$ which is defined by the rule $D_\alpha(f) = \alpha f'$ is the differentiation.

Proof. The linearity of the map D_α follows from the linearity of derivative of the polynomials. So the fact that D_α is differentiation is enough to verify for monomials.

$$D_\alpha(lx^n, l'x^m) = \begin{cases} \alpha(n+m)(l,l')x^{n+m-1}, & \text{if } n+m<p; \\ 0, & \text{if } n+m \geq p. \end{cases}$$

$$(D_\alpha(lx^n), l'x^m) + (lx^n, D(l'x^m)) = \alpha n(lx^{n-1}, l'x^m) +$$

$$+ \quad m\alpha(lx^n, l'x^{m-1}) = \begin{cases} \alpha(n+m)(l,l')x^{n+m-1}, & \text{if } n+m-1<p; \\ 0, & \text{if } n+m-1 \geq p. \end{cases}$$

Notice that if the degree $n + m = p$, then by definition (1) of the Lie bracket in Lie algebra $Fun(L_1, L)$ holds $n + m = 0$. Thus the upper equality coincides with the lower one and D_α is a differentiation. □

Therefore we can define a map φ from Lie algebra L_1 to the algebra of differentiations $\mathcal{D}(Fun(L_1, L))$ given by the rule $\alpha \mapsto D_\alpha$, where $D_\alpha(f) = \alpha f'$. The map φ is a homomorphism. Really, $\varphi((\alpha, \beta)) = 0$ and $D_\alpha D_\beta(f) - D_\beta D_\alpha(f) = \alpha\beta f'' - \beta\alpha f'' = 0$.

Definition 2. The semidirect product of Lie algebra L_1 with Lie algebra $Fun(L_1, L)$, which corresponds to the homomorphism φ, we call the *wreath product of Lie algebra L with L_1* and denote by $L \wr L_1$.

Thus, $L \wr L_1 := L_1 \leftthreetimes_\varphi Fun(L_1, L) = \{[a, f] | \; a \in L_1, f \in Fun(L_1, L)\}$ with Lie bracket

$$([a_1, f_1], [a_2, f_2]) = [0, a_1\frac{\partial f_2}{\partial x} - a_2\frac{\partial f_1}{\partial x} + (f_1, f_2)]. \tag{2}$$

Remark 1. Definition 2 allows us to consider the wreath product $L \wr \; L_1 \wr \; \ldots \wr \; L_1$ for an arbitrary Lie algebra L.

The subset of elements $[a, e]$ of $L \wr L_1$ forms the subalgebra P, which is isomorphic to L_1. The subset H of elements $[0, f]$ is a subalgebra of $L \wr L_1$ which is isomorphic to $Fun(L_1, L)$.

Proposition 3. Let L be a solvable Lie algebra of the derived length n. Then $L \wr L_1$ is solvable of the derived length $n+1$.

Proof. By the definition of the Lie bracket in Lie algebra $Fun(L_1, L)$ the coefficients of a polynomial (f, g), $f, g \in Fun(L_1, L)$, belong to the algebra $L^{(1)} = (L, L)$. Thus the inclusion $\big(Fun(L_1, L), Fun(L_1, L)\big) \subseteq Fun(L_1, L^{(1)})$ holds.

The following inclusion $(L \wr L_1)^{(1)} \subset [0, Fun(L_1, L)]$ is also correct. Thus we have

$$\begin{aligned}\big([0, Fun(L_1, L)], [0, Fun(L_1, L)]\big) &= [0, \big(Fun(L_1, L), Fun(L_1, L)\big)] \subseteq \\ &\subseteq [0, Fun(L_1, L^{(1)})].\end{aligned}$$

Thus, $(L \wr L_1)^{(2)} \subseteq [0, Fun(L_1, L^{(1)})]$. If we continue this process we obtain that

$$(L \wr L_1)^{(n+1)} \subseteq [0, Fun(L_1, L^{(n)})].$$

Thus, if L is solvable of derived length n then $L \wr L_1$ is solvable of derived length at most $n+1$.

Notice that $[0, L]$ is contained in $(L \wr L_1)^{(1)}$, where we consider elements of L as constant polynomials. Thus

$$[0, L^{(n-1)}] \subseteq (L \wr L_1)^{(n)}.$$

From this follows that $L \wr L_1$ is solvable of derived length at least $n+1$. Thus $L \wr L_1$ is solvable of derived length n. □

Proposition 4. Let L be a nilpotent Lie algebra of nilpotent class n. Then $L \wr L_1$ is nilpotent of nilpotent class np.

Proof. Consider the lower central series of the Lie algebra $L \wr L_1$. Let $\gamma_0 = L \wr L_1$, $\gamma_k = (\gamma_{k-1}, L \wr L_1)$ be the k-th term of the lower central series.

Denote $F_k = \{f |\ [0, f] \in \gamma_k\} \subset Fun(L_1, L)$. Then $\gamma_k = [0, F_k]$. From formula (2) follows that every polynomial $f \in F_k$ has monomials of degree $\le p - 1 - k$ with coefficients from L and f has also monomials of degree $\le p - 1$ with coefficients from $\gamma_1(L)$.

Hence, $F_p \subset Fun(L_1, \gamma_1(L))$. Notice, that polynomials of F_p have monomials of degree $\le p - 1$ with coefficients from $\gamma_1(L)$, $\gamma_2(L)$, $\ldots$, $\gamma_p(L)$.

In a similar we obtain $F_{p+p} \subset Fun(L_1, \gamma_2(L))$ and so on. Thus,

$$\gamma_{p \cdot n} = [0, F_{p \cdot n}] \subset [0, Fun(L_1, \gamma_n(L))] = [0, 0].$$

Thus, if L is nilpotent of nilpotent class n then $L \wr L_1$ is nilpotent of nilpotent class at most np.

Notice that $[0, L] \subseteq \gamma_k(L \wr L_1)$, $1 \leq k \leq p-1$. In a similar way we obtain $[0, \gamma_1(L)] \subseteq \gamma_l(L \wr L_1)$, $p \leq l \leq 2p-1$.

Thus, $[0, \gamma_{(n-1)}(L)] \subseteq \gamma_s(L \wr L_1)$, $(n-1)p \leq s \leq np-1$. Consequently, Lie algebra $L \wr L_1$ is nilpotent of nilpotent class at least np.

Thus $L \wr L_1$ is nilpotent of nilpotent class np. □

3. Lie algebras associated with the Sylow p-subgroups of symmetric groups

We will consider the notion of "tableau" introduced by L. Kaloujnine in [4]. On the set of all tableaux of the length m over $\mathbb{F}_p$ we introduce the structure of Lie algebra in the following way. Define the addition, Lie bracket (,) and the multiplication on the elements of $\mathbb{F}_p$ for tableaux

$$u = [u_1, u_2(x_1), u_3(x_1, x_2), \ldots], \; v = [v_1, v_2(x_1), v_3(x_1, x_2), \ldots]$$

by the following equalities ($1 \leq k \leq m$):

$$\begin{aligned} (i) \quad & \{u + v\}_k = u_k + v_k; \\ (ii) \quad & \{(u, v)\}_k = \sum_{i=1}^{k-1} \left(\frac{\partial v_k}{\partial x_i} \cdot u_i - v_i \cdot \frac{\partial u_k}{\partial x_i}\right); \\ (iii) \quad & \{\alpha \cdot u\}_k = \alpha \cdot u_k, \alpha \in \mathbb{F}_p. \end{aligned}$$

where $u_1 = a_1 \in \mathbb{F}_p$,

$$u_k = a_k(x_1, x_2, \ldots, x_{k-1}) = a_k(\overline{x}_{k-1}) \in \mathbb{F}_p[x_1 \ldots, x_{k-1}]/I_{k-1},$$

where I_{k-1} is an ideal, generated by polynomials $x_1^p, x_2^p, \ldots, x_{k-1}^p$.

According to [9] the set of all tableaux over $\mathbb{F}_p$ with operations (i) – (iii) forms the Lie algebra denoted by L_m.

Denote by $L(P_m)$ the Lie algebra associated with the lower central series of the Sylow p-subgroup P_m of the symmetric group S_{p^m}. The structure of Lie algebra $L(P_m)$ was investigated in [9]. In particular, the following theorem was proved:

Theorem 5. Lie algebra $L(P_m)$ is isomorphic to the algebra L_m.

The following theorem holds:

Theorem 6. $L_m \simeq L_1 \wr L_1 \wr \ldots \wr L_1$.

Proof. Note that since $P_m \simeq C_p \wr C_p \wr \ldots \wr C_p$, and Lie algebra $L_m \simeq L(P_m)$, then we can replace the assertion of the theorem by $L(C_p \wr C_p \wr \ldots \wr C_p) \simeq L_1 \wr L_1 \wr \ldots \wr L_1$.

We will prove the theorem by induction on the number of the components of the wreath product. Define

$$P_n = \underbrace{C_p \wr \ldots \wr C_p}_{n} \text{ and } \mathcal{L}_n = \underbrace{L_1 \wr \ldots \wr L_1}_{n}.$$

If $n = 1$ then $L(C_p) \simeq L_1$ and the assertion is correct. Assume that the assertion is true for n, that is $L(P_n) \simeq \mathcal{L}_n$. We will show that $\mathcal{L}_n \wr L_1 \simeq L(P_n \wr C_p)$.

Every function $f : L_1 \to \mathcal{L}_n$ can be uniquely represented by the tableau

$$[a_1(x_1), a_2(x_1, x_2), \ldots, a_n(x_1, \ldots, x_n)], \tag{3}$$

where $a_k(x_1, \ldots, x_k) \in \mathbb{F}_p[x_1, \ldots, x_k]/I_k$. Really, $f(x_1) = l_{p-1}x_1^{p-1} + \ldots + l_0$, where $l_i \in \mathcal{L}_n$ and according to the assumption of induction and theorem 5 $l_i = [b_0^i, b_1^i(x_2), \ldots, b_{n-1}^i(x_2, \ldots, x_n)]$. Then $f(x)$ is uniquely represented in the form $[a_1(x_1), a_2(x_1, x_2), \ldots, a_n(x_1, \ldots, x_n)]$, where

$$a_{i+1}(x_1, \ldots, x_{i+1}) = b_i^{p-1}(x_2, \ldots, x_{i+1})x_1^{p-1} + \ldots + b_i^0(x_2, \ldots, x_{i+1}),$$
$$i = 0, \ldots, n-1.$$

Then f' is represented in the form

$$\begin{aligned} f' &= (p-1)l_{p-1}x_1^{p-2} + \ldots + l_1 = \\ &= (p-1)[b_0^{p-1}, \ldots, b_{n-1}^{p-1}(x_2, \ldots, x_n)]x_1^{p-2} + \ldots \\ &\quad \ldots + [b_0^1, \ldots, b_{n-1}^1(x_2, \ldots, x_n)] = \\ &= [(p-1)b_0^{p-1}x_1^{p-2} + \ldots + b_0^1, \ldots, (p-1)b_{n-1}^{p-1}x_1^{p-2} + \ldots + b_{n-1}^1] = \\ &= [\frac{\partial}{\partial x_1}a_1(x_1), \frac{\partial}{\partial x_1}a_2(x_1, x_2), \ldots, \frac{\partial}{\partial x_1}a_n(x_1 \ldots, x_n)]. \end{aligned} \tag{4}$$

Moreover, for every functions $f = [a_1(x_1), a_2(x_1, x_2), \ldots, a_n(\overline{x}_n)]$, $g = [b_1(x_1), b_2(x_1, x_2), \ldots, b_n(\overline{x}_n)]$ the function (f, g) is of the form $[0, c_2(x_1, x_2), \ldots, c_n(x_1, \ldots, x_n)]$, where

$$c_k(x_1, \ldots, x_k) = \sum_{i=1}^{k-1}(a_i \frac{\partial}{\partial x_{i+1}}b_k - b_i \frac{\partial}{\partial x_{i+1}}a_k). \tag{5}$$

Indeed, from the linearity of representation (3) follows that it is enough to verify (5) only for monomials. Let $f = lx_1^m$ and $g = hx_1^k$, where $l =$

$[l_1, l_2(x_2), \ldots, l_n(x_2, \ldots, x_n)]$, $h = [h_1, h_2(x_2), \ldots, h_n(x_2, \ldots, x_n)] \in \mathcal{L}_n$. Then

$$\begin{aligned} f &= [l_0 x_1^m, l_1(x_2) x_1^m, \ldots, l_n(x_2, \ldots, x_n) x_1^m], \\ g &= [h_0 x_1^k, h_1(x_2) x_1^k, \ldots, h_n(x_2, \ldots, x_n) x_1^k] \end{aligned}$$

Then the coefficients from (5) look like:

$$\begin{aligned} c_j(x_1, \ldots, x_j) &= \sum_{i=1}^{j-1} (l_i x_1^m \frac{\partial}{\partial x_{i+1}} h_j x_1^k - h_i x_1^k \frac{\partial}{\partial x_{i+1}} l_j x_1^m) = \\ &= \begin{cases} \sum_{i=1}^{j-1} (l_i \frac{\partial}{\partial x_{i+1}} h_j - h_i \frac{\partial}{\partial x_{i+1}} l_j) x_1^{m+k}, & \text{if } m+k < p.; \\ 0, & \text{if } m+k \geq p. \end{cases} \end{aligned}$$

Let us write down how (f, g) is represented by the tableau (3):

$$\begin{aligned} (f, g) &= \begin{cases} (l, h) x_1^{m+k}, & \text{if } m+k < p; \\ 0, & \text{if } m+k \geq p. \end{cases} = \\ &= \begin{cases} [0, d_2(x_2), \ldots, d_n(x_2, \ldots, x_n)] x_1^{m+k}, & \text{if } m+k < p; \\ 0, & \text{if } m+k \geq p. \end{cases} = \\ &= \begin{cases} [0, d_2(x_2) x_1^{m+k}, \ldots, d_n(x_2, \ldots, x_n) x_1^{m+k}], & \text{if } m+k < p; \\ 0, & \text{if } m+k \geq p, \end{cases} \end{aligned}$$

$$\text{where } \; d_j(x_2, \ldots, x_j) = \sum_{i=1}^{j-1} (l_i \frac{\partial}{\partial x_{i+1}} h_j - h_i \frac{\partial}{\partial x_{i+1}} l_j).$$

Thus the function (f, g) is of the form $[0, c_2(x_1, x_2), \ldots, c_n(x_1, \ldots, x_n)]$.

Let us construct the map $\psi : \mathcal{L}_n \wr L_1 \to L(P_n \wr C_p)$ by the rule $\psi([a_0, f]) = [a_0, a_1(x_1), \ldots, a_n(x_1, \ldots, x_n)]$. According to proposition 1 and theorem 5 the map ψ is a bijection. Let us show that ψ is linear. Really:

$$\begin{aligned} &\psi(\alpha[a_0, f] + \beta[b_0, g]) = \psi([\alpha a_0 + \beta b_0, \alpha f + \beta g]) = \\ = \; &[\alpha a_0 + \beta b_0, \alpha a_1(x_1) + \beta b_1(x_1), \ldots, \alpha a_n(\overline{x}_n) + \beta b_n(\overline{x}_n)] = \\ = \; &\alpha[a_0, \ldots, a_n(\overline{x}_n)] + \beta[b_0, \ldots, b_n(\overline{x}_n)] = \alpha\psi([a_0, f]) + \beta\psi([b_0, g]). \end{aligned}$$

It remains to prove that $\psi(([a_0, f], [b_0, g])) = (\psi([a_0, f]), \psi([b_0, g]))$.

From (4) and (5) follows:

$$\begin{aligned} \psi(([a_0, f], [b_0, g])) &= \psi([0, a_0 g' - b_0 f' + (f, g)]) = \\ &= [0, d_1(x_1), \ldots, d_n(x_1, \ldots, x_n)], \text{ where} \end{aligned}$$

$$\begin{aligned} d_k &= a_0\frac{\partial}{\partial x_1}b_k - b_0\frac{\partial}{\partial x_1}a_k + \sum_{i=1}^{k-1}(a_i\frac{\partial}{\partial x_{i+1}}b_k - b_i\frac{\partial}{\partial x_{i+1}}a_k) = \\ &= \sum_{i=0}^{k-1}(a_i\frac{\partial}{\partial x_{i+1}}b_k - b_i\frac{\partial}{\partial x_{i+1}}a_k). \end{aligned}$$

Thus,

$$\begin{aligned} \psi(([a_0,f],[b_0,g])) &= ([a_0,a_1(x_1),\ldots,a_n(\overline{x}_n)],[b_0,b_1(x_1),\ldots,b_n(\overline{x}_n)]) = \\ &= (\psi([a_0,f]),\psi([b_0,g])). \end{aligned}$$

□

Let S_n be the group of all permutations of the set of n elements, where

$$n = a_0 + a_1p + a_2p^2 + \ldots + a_kp^k.$$

We describe the Lie algebra $L(Syl_p(S_n))$ associated with the Sylow p-subgroup of any symmetric group S_n in terms of wreath product of one-dimensional Lie algebras. It is well known (see [7]), that the Sylow p-subgroup of the symmetric group S_n is isomorphic to

$$Syl_p(S_n) \simeq \bigoplus_{l=0}^{k} \underbrace{Syl_p(S_{p^l}) \times \ldots \times Syl_p(S_{p^l})}_{a_l} \tag{6}$$

Proposition 7. Let $G = H \times K$ and $\gamma_i(H), \gamma_i(K)$ be the i-th terms of the lower central series of the groups H and K correspondingly. Then $\gamma_i(G) = \gamma_i(H) \times \gamma_i(K)$.

Proof. We will prove this assertion by induction. If $n = 0$ we have $\gamma_0(H) = H$, $\gamma_0(K) = K$ and $\gamma_0(G) = G = H \times K = \gamma_0(H) \times \gamma_0(K)$. Assume that the assertion is true for i, that is $\gamma_i(G) = \gamma_i(H) \times \gamma_i(K)$. Then

$$\begin{aligned} \gamma_{i+1}(G) &= [\gamma_i(G), G] = [\gamma_i(H) \times \gamma_i(K), H \times K] = \\ &= [\gamma_i(H), H] \times [\gamma_i(K), K] = \gamma_{i+1}(H) \times \gamma_{i+1}(K). \end{aligned}$$

Hence, we obtain $\gamma_i(G) = \gamma_i(H) \times \gamma_i(K)$ by induction on i, as required.

□

Corollary 8. $L(G) = L(H) \oplus L(K)$.

Proof. Recall, that Lie algebra associated with the lower central series of the group G (see [10])) is $L(G) = \bigoplus_{i=1}^{\infty} \gamma_i(G)/\gamma_{i+1}(G)$, where $\gamma_i(G)$ is i-th term of the lower central series of group G. Thus, we have

$$\begin{aligned} L(G) &= L(H \times K) = \oplus_{i\geq 0}\gamma_i(G)/\gamma_{i+1}(G) = \\ &= \oplus_{i\geq 0}(\gamma_i(H) \times \gamma_i(K))/(\gamma_{i+1}(H) \times \gamma_{i+1}(K)) = \\ &= \oplus_{i\geq 0}\gamma_i(H)/\gamma_{i+1}(H) \oplus_{i\geq 0} \gamma_i(K)/\gamma_{i+1}(K) = L(H) \oplus L(K) \end{aligned}$$

□

Theorem 9. Lie algebra associated with the Sylow p-subgroup of the group S_n is isomorphic to

$$L(Syl_p(S_n)) \simeq \oplus_{r=0}^{k} \underbrace{L(Syl_p(S_{p^r})) \oplus \ldots \oplus L(Syl_p(S_{p^r}))}_{a_r}$$

Proof. The assertion of the theorem directly follows from (6) and corollary (8). □

Remark 2. According to the theorem 5 we can write down the assertion of the theorem in the form

$$L(Syl_p(S_n)) \simeq \oplus_{r=0}^{k}\big(\oplus_{i=1}^{a_r} \wr_{j=1}^{r} L_1\big)$$

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Received by the editors: 27.03.2005
and final form in 05.04.2005.