On the group of extensions for the bicrossed product construction for a locally compact group

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ABSTRACT. For the cocycle bicrossed product construction applied to a locally compact group and its two subgroups, we give a simple description of the group of the corresponding extensions in terms of the second cohomology group of a certain complex of continuous functions on the group. Using this description, we find pairs of continuous cocycles for two subgroups of the Heisenberg group.

Introduction

A method for constructing nontrivial examples of finite ring groups, now known as finite Kac algebras, was proposed in [1]. It consists in using two subgroups of a group satisfying certain conditions for constructing a commutative algebra of functions on one subgroup and then extending it to a nontrivial Kac algebra, via a pair of cocycles, with the group algebra of the other subgroup. This method is now called the bicrossed product construction and was generalized to bialgebras in [2] and to locally compact quantum groups in [3].

For fixed subgroups, the set of extensions forms a group whose elements are determined by equivalence classes of pairs of cocycles. This group formed by equivalence classes of pairs of measurable functions is

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described in [4] for a locally compact group in terms of the second coho-
ology group of a certain complex constructed from a bicomplex.

The purpose of this paper is to consider the case where the exten-
sions are obtained using continuous cocycles and to give a simple direct
construction of a complex such that its second cohomology group is iso-
morphic to the group of such extensions. This is done in Section 1 after
first making necessary definitions. In Section 2 we use this result and
the ideas from [5] to construct pairs of cocycles for two subgroups of the
Heisenberg group.

1. Definitions and the main result

**Definition 1** ([6]). Let $K$ be a locally compact group, $G$, $H$ subgroups
of $K$ satisfying the conditions

$$G \cdot H = K \quad \text{and} \quad G \cap H = \{e\}.$$ 

Then $(G, H)$ is called a matched pair of locally compact groups.

**Remark 1.** For a more general definition of a matched pair of locally
compact groups, see [3].

In what follows, $g$, $h$, $k$ with, possibly, subscripts denote elements of
the groups $G$, $H$, $K$, respectively.

Let $(G, H)$ be a matched pair of locally compact groups. It is
known [3] that there are right and left actions, $\triangleleft : H \times G \to H$ and
$\triangleright : H \times G \to G$, given by

$$(h \triangleleft g) \cdot g = (h \triangleright g) \cdot (h \triangleleft g)$$

and satisfying

$$h \cdot g = (h \triangleright g) \cdot (h \triangleleft g)$$

and

$$(h_1 h_2) \triangleleft g = (h_1 \triangleleft (h_2 \triangleright g))(h_2 \triangleleft g),$$

$h \triangleright (g_1 g_2) = (h \triangleright g_1)((h \triangleleft g_1) \triangleright g_2)$. (1)

Denote $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and let $\mathbb{T}$ be the Abelian group $\mathbb{R}/2\pi \mathbb{Z}$. 

**Definition 2** ([3]). Let $(G, H)$ be a matched pair of locally compact
groups. A pair of continuous maps $(u, v)$, $u : H \times G \times G \to \mathbb{T}$, $v : H \times H \times G \to \mathbb{T}$ is called a pair of cocycles for the pair $(G, H)$ if the
following identities hold:

$$u(h \triangleleft g_1, g_2, g_3)u(h, g_1, g_2 g_3) = u(h, g_1, g_2)u(h, g_1 g_2, g_3), \quad (2)$$

$$v(h_1, h_2, h_3 \triangleright g)v(h_1 h_2, h_3, g) = v(h_1, h_2 h_3, g)v(h_2, h_3, g), \quad (3)$$

$$v(h_1, h_2, 1 g_2)u(h_1 h_2, g_1, g_2) = v(h_1, h_2, g_1)u(h_2, g_1, g_2)$$

$$\cdot v(h_1 \triangleleft (h_2 \triangleright g_1), h_2 \triangleleft g_1, g_2)$$

$$\cdot u(h_1, h_2 \triangleright g_1, (h_2 \triangleleft g_1) \triangleright g_2). \quad (4)$$
Two pairs of cocycles \((u_1, v_1)\) and \((u_2, v_2)\) for \((G, H)\) are called equivalent if there exists a continuous function \(r : H \times G \to \mathbb{T}\) such that

\[
\begin{align*}
    u_1(h, g_1, g_2)u_2(h, g_1, g_2)^{-1} &= r(h, g_1)r(h \triangleleft g_1, g_2)r(h, g_1g_2)^{-1}, \\
    v_1(h_1, h_2, g)v_2(h_1, h_2, g)^{-1} &= r(h_1h_2, g)r(h_1, h_2 \triangleright g)^{-1}r(h_2, g)^{-1}.
\end{align*}
\]

We will denote the equivalence class of a pair \((u, v)\) by \([u, v]\). It is known [3] that the set of equivalence classes \([u, v]\) forms a group with respect to the operations \([u_1, v_1] \cdot [u_2, v_2] = [u_1u_2, v_1v_2]\), \([u, v]^{-1} = [u^{-1}, v^{-1}]\), and the identity element \([1, 1]\). We will consider the subgroup of this group formed by the classes \([u, v]\) such that \(u(h_1, g_2, g_3) = 1\) if at least one of the elements \(h_1, g_2, g_3\) is the identity of the corresponding group and the same holds for \(v\). This subgroup will be denoted by \(H^2_0(\text{m.p., } \mathbb{T})\).

Let \((G, H)\) be a matched pair of locally compact groups. Denote by \(\mathcal{C}^n_H(K, \mathbb{T})\), or simply by \(\mathcal{C}^n_H(K)\), \(n = 0, 1, \ldots\), the set of continuous functions \(c^n : K^{n+1} \to \mathbb{T}\) such that

\[
\begin{align*}
    c^n(gk_1, k_2, \ldots, k_n, k_{n+1}h) &= c^n(k_1, k_2, \ldots, k_n, k_{n+1}), \\
    c^n(k_1, \ldots, k_{j-1}, e_K, k_{j+1}, \ldots, k_{n+1}) &= 0
\end{align*}
\]

for all \(g \in G, h \in H, k_j \in K, j = 1, \ldots, n + 1\). Here \(e_K\) denotes the identity element in \(K\). \(\mathcal{C}^n_H\) is an Abelian group with respect to the pointwise addition. As usual [7], define

\[
(d^n c^n)(k_1, \ldots, k_{n+2}) = c^n(k_2, \ldots, k_{n+2}) + \sum_{j=1}^{n+1} (-1)^j c^n(k_1, \ldots, k_j k_{j+1}, \ldots, k_{n+2}) + (-1)^{n+2} c^n(k_1, \ldots, k_{n+1})
\]

and thus obtain the following complex:

\[
\begin{align*}
    \mathcal{C}^0_H(K) &\xrightarrow{d^0} \mathcal{C}^1_H(K) &\xrightarrow{d^1} \mathcal{C}^2_H(K) &\xrightarrow{d^2} \mathcal{C}^3_H(K) &\xrightarrow{d^3} \cdots,
\end{align*}
\]

where, as easily seen, \(\mathcal{C}^0_H(K)\) can be identified with the group that has the only element 0 and \(\mathcal{C}^1_H(K)\) with the group of all continuous functions \(c^1\) on \(K\) satisfying \(c^1|_G = c^1|_H = 0\).

**Theorem.** Let \(\mathcal{H}^2_H(K) = \ker d^2/\text{Im } d^1\) denote the second cohomology group of complex (9). Then the map \(\theta : H^2_0(\text{m.p., } \mathbb{T}) \to \mathcal{H}^2_H(K)\) defined by

\[
\theta([u, v])(k_1, k_2, k_3) = \frac{1}{i} \ln u(h_1, g_2, h_2 \triangleright g_3) + \frac{1}{i} \ln v(h_1 \triangleleft g_2, h_2, g_3),
\]

where \(u, v\) are continuous functions satisfying the conditions of the theorem.
is a group isomorphism. Here \( \ln : \mathbb{T} \to i \mathbb{T} \) is the principle branch of the logarithm, \( k_j = g_j h_j, \ j = 1, 2, 3, \) and \( i = \sqrt{-1} \).

The proof of the theorem will be divided into several lemmas. But before, let us denote \( \hat{u} = \frac{1}{i} \ln u, \hat{v} = \frac{1}{i} \ln v, \) and \( \hat{r} = \frac{1}{i} \ln r. \) With these notations, the map (10) becomes

\[
\theta([u,v])(k_1, k_2, k_3) = \hat{u}(h_1, g_2, h_2 \triangleright g_3) + \hat{v}(h_1 \triangleleft g_2, h_2, g_3),
\]

the defining relations (2), (3), (4) will read

\[
\begin{align*}
\hat{u}(h \triangleleft g_1, g_2, g_3) - \hat{u}(h, g_1 g_2, g_3) + \hat{u}(h, g_1, g_2 g_3) - \hat{u}(h, g_1, g_2) &= 0, \\
\hat{v}(h_2, h_3, g) - \hat{v}(h_1 h_2, h_3, g) + \hat{v}(h_1, h_2 h_3, g) - \hat{v}(h_1, h_2, h_3 \triangleright g) &= 0,
\end{align*}
\]

and the equivalence relations (5) are

\[
\begin{align*}
\hat{u}_1(h, g_1, g_2) - \hat{u}_2(h, g_1, g_2) &= \hat{r}(h, g_1) + \hat{r}(h \triangleleft g_1, g_2) - \hat{r}(h, g_1 g_2), \\
\hat{v}_1(h_1, h_2, g) - \hat{v}_2(h_1, h_2, g) &= \hat{r}(h_1 h_2, g) - \hat{r}(h_1, h_2 \triangleright g) - \hat{r}(h_2, g).
\end{align*}
\]

**Lemma 1.** Let a pair \((\hat{u}, \hat{v})\) satisfy (12), (13), (14), and

\[
f(k_1, k_2, k_3) = \hat{u}(h_1, g_2, h_2 \triangleright g_3) + \hat{v}(h_1 \triangleleft g_2, h_2, g_3),
\]

\( k_j = g_j h_j, \ j = 1, 2, 3. \) Then \( f \) is a 2-cocycle for the complex (9).

**Proof.** Indeed, by the definition of \( f, \) for \( k_j = g_j h_j, \ j = 1, \ldots, 4, \) we have

\[
(d^2 f)(k_1, k_2, k_3, k_4)
= f(k_2, k_3, k_4) - f(k_1 k_2, k_3, k_4) + f(k_1, k_2 k_3, k_4)
\]

\[
- f(k_1, k_2, k_3 k_4) + f(k_1, k_2, k_3)
= f(g_2 h_2, g_3 h_3, g_4 h_4) - f(g_1 h_1 \triangleright g_2)(h_1 \triangleleft g_2) h_2, g_3 h_3, g_4 h_4)
\]

\[
+ f(g_1 h_1, g_2 h_2 \triangleright g_3)(h_2 \triangleleft g_3) h_3, g_4 h_4)
- f(g_1 h_1, g_2 h_2, g_3 (h_3 \triangleright g_4) h_4) + f(g_1 h_1, g_2 h_2, g_3 h_3)
\]

\[
= \hat{u}(h_2, g_3, h_3 \triangleright g_4) + \hat{v}(h_2 \triangleleft g_3, h_3, g_4) - \hat{u}((h_1 \triangleleft g_2) h_2, g_3, h_3 \triangleright g_4)
\]

\[
- \hat{v}(((h_1 \triangleleft g_2) h_2) \triangleleft g_3, h_3, g_4) + \hat{u}(h_1, g_2 (h_2 \triangleright g_3), (h_2 \triangleleft g_3) h_3 \triangleright g_4)
\]

\[
+ \hat{v}(h_1 \triangleleft (g_2 (h_2 \triangleright g_3)), (h_2 \triangleleft g_3) h_3, g_4) - \hat{u}(h_1, g_2, h_2 \triangleright (g_3 (h_3 \triangleright g_4)))
\]

\[
- \hat{v}(h_1 \triangleleft g_2, h_2, g_3 (h_3 \triangleright g_4)) + \hat{u}(h_1, g_2, h_2 \triangleright g_3) + \hat{v}(h_1 \triangleleft g_2, h_2, g_3).
\]
Replacing \( h_1, g_1, \) and \( g_2 \) in (14) with \( h_1 \triangleleft g_2, g_3, \) and \( h_3 \triangleright g_4, \) respectively, we get that

\[
\tilde{u}((h_1 \triangleleft g_2)h_2, g_3, h_3 \triangleright g_4) + \tilde{v}(h_1 \triangleleft g_2, h_2, g_3(h_3 \triangleright g_4)) = \tilde{u}(h_2, g_3, h_3 \triangleright g_4) + \tilde{v}(h_1 \triangleleft g_2, h_2 \triangleright g_3, (h_2 \triangleleft g_3) \triangleright (h_3 \triangleright g_4)) + \tilde{v}(h_1 \triangleleft g_2, h_2, g_3).
\]

On the other hand, extending \( \tilde{r} \) from \( H \times G \to \overline{T} \) to \( K \times K \to \overline{T} \) by setting \( \tilde{r}(g_1 h_1, g_2 h_2) = \tilde{r}(h_1, g_2), \) we have

\[
(d^1 \tilde{r})(k_1, k_2, k_3) = \tilde{r}(k_2, k_3) - \tilde{r}(k_1 k_2, k_3) = \tilde{r}(k_1, k_2) + \tilde{r}((h_1 \triangleleft g_2)h_2, g_3) + \tilde{r}(h_1, g_2(h_2 \triangleright g_3)) - \tilde{r}(h_1, g_2).
\]

Comparing the above, we see that \( f_1 = d^1(-r). \)

\[ \square \]

Corollary 1. The map \( \theta \) is well-defined.

Lemma 3. Let \( f \in \mathcal{H}_u^2 \). Then \( f \) satisfies the following:

\[ f(k_1, k_2, k_3) = f(h_1, g_2, h_2 \triangleright g_3) + f(h_1 \triangleleft g_2, h_2, g_3). \]

Proof. Since \( f \) is a 2-cocycle,

\[
0 = (d^2 f)(h_1, g_2, h_2, g_3) = f(g_2, h_2, g_3) - f(h_1 g_2, h_2, g_3) + f(h_1, g_2 h_2, g_3) - f(h_1, g_2, h_2 g_3) = f(h_1, g_2, h_2, g_3).
\]

But \( f(g_2, h_2, g_3) = f(e_K, h_2, g_3) = 0, f(h_1 g_2, h_2, g_3) = f(h_1 \triangleleft g_2, h_2, g_3), f(h_1, g_2, h_2 g_3) = f(h_1, g_2, h_2 \triangleright g_3), \) and \( f(h_1, g_2, h_2) = f(h_1, g_2, e_K) = 0. \)

\[ \square \]

Corollary 2. The map \( \theta \) is a surjection.
Proof. For \( f \in \mathcal{H}_n^2 \), define \( \tilde{u}(h_1, g_2, g_3) = f(h_1, g_2, g_3) \) and \( \tilde{v}(h_1, h_2, g_3) = f(h_1, h_2, g_3) \). Then \( \tilde{u} \) and \( \tilde{v} \) satisfy (12), (13), and (14). \( \square \)

**Lemma 4.** The map \( \theta \) is an injection.

**Proof.** Indeed, let \( \theta([u, v])(k_1, k_2, k_3) = (d^1\tilde{r})(k_1, k_2, k_3) \), that is,

\[
\tilde{u}(h_1, g_2, h_2 \triangleright g_3) + \tilde{v}(h_1 \triangleleft g_2, h_2, g_3)
= \tilde{r}(k_2, k_3) - \tilde{r}(k_1 k_2, k_3) + \tilde{r}(k_1, k_2 k_3) - \tilde{r}(k_1, k_2)
= \tilde{r}(h_2, g_3) - \tilde{r}((h_1 \triangleleft g_2) h_2, g_3) + \tilde{r}(h_1, g_2 (h_2 \triangleright g_3)) - \tilde{r}(h_1, g_2).
\]

By setting \( h_2 = e_H \) and then \( g_2 = e_G \), we obtain

\[
\tilde{u}(h_1, g_2, g_3) = -\tilde{r}(h_1 \triangleleft g_2, g_3) + \tilde{r}(h_1, g_2 g_3) - \tilde{r}(h_1, g_2)
\tilde{v}(h_1, h_2, g_3) = \tilde{r}(h_2, g_3) - \tilde{r}(h_1 h_2, g_3) + \tilde{r}(h_1, h_2 \triangleright g_3),
\]

that is \([u, v] = [1, 1]\). \( \square \)

Now, Corollaries 1, 2 and Lemma 4 prove the theorem.

2. An example for the Heisenberg group

As follows from (16), to find the functions \( \tilde{u}, \tilde{v} \), we need to construct a 2-cocycle \( f \) of the complex (9). Note that \( f \) is, actually, a 3-cocycle of the group \( K \) [7] satisfying the additional relations (6), (7).

Thus we construct a 3-cocycle \( F \) for the corresponding matched pair of the Lie algebras, \((\mathfrak{g}, \mathfrak{h})\), by using Proposition 1 in [5], find nonequivalent 3-cocycles in terms of Proposition 2 in [5], and consider the corresponding left-invariant form \( \omega_F \) on \( K \). Following the procedure of finding a 3-cocycle on a Lie group from a 3-cocycle on the Lie algebra [7], we consider the 3-simplex \( \sigma \) given by

\[
\sigma(h_1, g_2 h_2, g_3) = (h_1 \triangleright (g_2 \cdot (h_2 \triangleright g_3^{p_3}))^{p_2})
= (h_1 \triangleleft (g_2 (h_2 \triangleright g_3^{p_2}))^{p_2} (h_2 \triangleleft g_3^{p_3} q_2)^{q_1})^{q_1}, \tag{17}
\]

where \( p_j, q_j : \Delta_3 \to \mathbb{R} \) are some differentiable functions on the standard 3-simplex \( \Delta_3 = \{(t_1, t_2, t_3) \in \mathbb{R}^3 : t_1, t_2, t_3 \geq 0, t_1 + t_2 + t_3 \leq 1\} \). Then the 2-cocycle \( f \) can be found in the form

\[
f(h_1, g_2 h_2, g_3) = \int_{\sigma(h_1, g_2 h_2, g_3)} \omega_F. \tag{18}
\]
Using the above procedure, we now construct pairs of continuous co-cycles \((\tilde{u}, \tilde{v})\) for the matched pair of Lie groups \((G, H)\) associated with the Heisenberg group. Denote

\[
g(\tilde{a}) = \begin{pmatrix}
1 & \tilde{a}^t & 0 \\
\tilde{a} & 1_n & \tilde{a}^t \\
0 & 0 & 1
\end{pmatrix}, \quad h(\tilde{x}, y) = \begin{pmatrix}
1 & \tilde{x}^t & y \\
\tilde{x} & 1_n & \tilde{y} \\
0 & 0 & 1
\end{pmatrix},
\]

where \(\tilde{a}, \tilde{x} \in \mathbb{R}^n, y \in \mathbb{R}, \tilde{x}^t\) is the transpose of \(\tilde{x}\), and \(1_n\) denotes the unit matrix on \(\mathbb{R}^n\). The groups \(G = \{g(\tilde{a}) : \tilde{a} \in \mathbb{R}^n\}\) and \(H = \{h(\tilde{x}, y) : \tilde{x} \in \mathbb{R}^n, y \in \mathbb{R}\}\) form a matched pair of Abelian Lie groups with the mutual actions

\[
h(\tilde{x}, y) \triangleleft g(\tilde{a}) = g(\tilde{a}), \quad h(\tilde{x}, y) \triangleright g(\tilde{a}) = h(\tilde{x}, y + \tilde{a} \cdot \tilde{x}),
\]

where \(\tilde{a} \cdot \tilde{x}\) is the scalar product of \(\tilde{a}, \tilde{x} \in \mathbb{R}^n\).

Thus the group \(K = GH\) is the Heisenberg group, that is, the group of matrices of the form

\[
\begin{pmatrix}
1 & \tilde{x}^t & y \\
\tilde{x} & 1_n & \tilde{y} \\
0 & 0 & 1
\end{pmatrix}.
\]

Consider the corresponding matched pair of the Abelian Lie algebras \((\mathfrak{g}, \mathfrak{h})\). The Lie algebras \(\mathfrak{g}\) and \(\mathfrak{h}\) consist of the matrices

\[
\begin{pmatrix}
0 & \tilde{a}^t & 0 \\
\tilde{a} & 0_n & \tilde{a}^t \\
0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & \tilde{x}^t & y \\
\tilde{x} & 0_n & \tilde{y} \\
0 & 0 & 0
\end{pmatrix},
\]

respectively. Let \(A_j \in \mathfrak{g}, j = 1, \ldots, n\), denote the matrix with \(\tilde{a}\) having 1 at the \(j\)th place and the rest 0. The matrices \(X_j\) and \(Y\) are defined similarly. Then \(A_j, X_j, Y\) form a basis in the Lie algebra \(\mathfrak{k}\) of the Lie group \(K\). The mutual actions are given by

\[
X_j \triangleleft A_k = \delta_{jk} Y, \quad Y \triangleleft A_j = X_j \triangleright A_k = Y \triangleright A_k = 0,
\]

where \(\delta_{jk}\) denotes Kronecker’s symbol.

Using Propositions 1, 2 in [5] we find that the functionals \(F_{jkl}^1\) and \(F_{jkl}^2\), defined on the basis of the Lie algebra \(\mathfrak{k}\) by

\[
F_{jkl}^1(X_j, X_k, A_l) = 1, \quad 1 \leq j < k \leq n, \quad l = 1, \ldots, n, \quad (19)
\]

and zero on other basis elements, and

\[
F_{jkl}^2(X_j, A_k, A_l) = 1, \quad j = 1, \ldots, n, \quad 1 \leq k < l \leq n, \quad j \neq k, \quad j \neq l, \quad (20)
\]
and zero otherwise, make a basis in the space of equivalence classes of 3-cocycles for the matched pair of Lie algebras \((g, h)\). Thus the dimension of the corresponding cohomology group of the matched pair of the Lie algebras is \(n(n - 1)^2\).

The left-invariant 3-forms on \(K\) that correspond to \(F_{jkl}^1\) and \(F_{jkl}^2\) are

\[
\omega_{jkl}^1 = dx^j \wedge dx^k \wedge da^l, \quad \omega_{jkl}^2 = dx^j \wedge da^k \wedge da^l,
\]

where \(x^j\) denotes the \(j\)-th coordinates of \(\vec{x} \in \mathbb{R}^n\).

Using (16), (17) and (18), we obtain the corresponding pairs of cocycles \(\tilde{u}, \tilde{v}\) for the matched pair \((G, H)\),

\[
\tilde{u}_{jkl}^1(h(\vec{x}, y), g(\vec{a}_1), g(\vec{a}_2)) = 0, \\
\tilde{v}_{jkl}^1(h(\vec{x}_1, y_1), h(\vec{x}_2, y_2), g(\vec{a})) = a^l \begin{vmatrix} x^j_1 & x^j_2 \\ x^k_1 & x^k_2 \end{vmatrix}, \tag{21}
\]

where \(j < k, l = 1, \ldots, n\), and

\[
\tilde{u}_{jkl}^2(h(\vec{x}, y), g(\vec{a}_1), g(\vec{a}_2)) = x^j \begin{vmatrix} a^k_1 & a^k_2 \\ a^l_1 & a^l_2 \end{vmatrix}, \\
\tilde{v}_{jkl}^2(h(\vec{x}_1, y_1), h(\vec{x}_2, y_2), g(\vec{a})) = 0, \tag{22}
\]

for \(j < k, j \neq k, j \neq l\).

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