On wildness of idempotent generated algebras associated with extended Dynkin diagrams

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Abstract. Let $\Lambda$ denote an extended Dynkin diagram with vertex set $\Lambda_0 = \{0, 1, \ldots, n\}$. For a vertex $i$, denote by $S(i)$ the set of vertices $j$ such that there is an edge joining $i$ and $j$; one assumes the diagram has a unique vertex $p$, say $p = 0$, with $|S(p)| = 3$. Further, denote by $\Lambda \setminus 0$ the full subgraph of $\Lambda$ with vertex set $\Lambda_0 \setminus \{0\}$. Let $\Delta = (\delta_i \mid i \in \Lambda_0) \in \mathbb{Z}^{|\Lambda_0|}$ be an imaginary root of $\Lambda$, and let $k$ be a field of arbitrary characteristic (with unit element 1). We prove that if $\Lambda$ is an extended Dynkin diagram of type $\tilde{D}_4$, $\tilde{E}_6$ or $\tilde{E}_7$, then the $k$-algebra $Q_k(\Lambda, \Delta)$ with generators $e_i$, $i \in \Lambda_0 \setminus \{0\}$, and relations $e_i^2 = e_i$, $e_i e_j = 0$ if $i$ and $j \neq i$ belong to the same connected component of $\Lambda \setminus 0$, and $\sum_{i=1}^{n} \delta_i e_i = \delta_0 1$ has wild representation type.

1. Formulation of the main result

Throughout the paper, we keep the right-side notation. By $k$ we will denote a fixed field of arbitrary characteristic; for a natural number $n$ and $1 \in k$, we identify $n1$ with $n$.

Let $\Lambda$ be an nonoriented graph without loops and multiple edges, and let $i$ be a vertex of $\Lambda$. Denote by $S(i)$ the set of vertices $j$ such that there is an edge joining $i$ and $j$. The vertex $i$ is said to be outer if $|S(i)| \leq 1$, inner if $|S(i)| > 1$, weakly inner if $|S(i)| = 2$ and strongly inner if $|S(i)| > 2$.

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Now let $\Lambda$ be a finite connected tree with vertex set $\Lambda_0 = \{0, 1, \ldots, n\}$. We assume that 0 is the unique strongly inner vertex, and denote by $\Lambda \setminus 0$ the full subgraph of $\Lambda$ with vertex set $\Lambda_0 \setminus \{0\}$. Given a vector $P = (p_0, p_1, \ldots, p_n) \in \mathbb{Z}^{1+n}$, we denote by $Q_k(\Lambda, P)$ the $k$-algebra with generators $e_i$, $1 \leq i \leq n$, and relations

1) $e_i^2 = e_i$ ($1 \leq i \leq n$);
2) $e_i e_j = 0$ if $i$ and $j \neq i$ belong to the same connected component of $\Lambda \setminus 0$;
3) $\sum_{i=1}^n p_i e_i = p_0$.

In this paper we study finite-dimensional representations of the algebra $Q_k(\Lambda, P)$ with $\Lambda$ being an extended Dynkin diagram. What we consider here is concerned with Yu. S. Samoilenko’s investigations [1].

Before we formulate the main results of this paper, we recall some definitions.

Let $\Lambda$ and $\Gamma$ be algebras over a field $k$. A matrix representation of $\Lambda$ over $\Gamma$ is a homomorphism $\varphi : \Lambda \to \Gamma^{s \times s}$ of algebras, where $s$ is a natural number and $\Gamma^{s \times s}$ the set of all $s \times s$-matrices with entries in $\Gamma$. Two representations $\varphi$ and $\psi$ of $\Lambda$ over $\Gamma$ are called equivalent if $\deg \varphi = \deg \psi$ and there exists an invertible matrix $\alpha$, with entries in $\Gamma$, such that $\varphi(\lambda)\alpha = \alpha \psi(\lambda)$ for every $\lambda \in \Lambda$. The indecomposability and direct sum of representations are defined in a natural way.

Let $\Lambda$ be a $k$-algebra, and $\Sigma = k(x, y)$ be the free associative $k$-algebra in two noncommuting variables $x$ and $y$. A representation $\gamma$ of $\Lambda$ over $\Sigma$ is said to be strict if it satisfies the following conditions:

1) the representation $\gamma \otimes \varphi$ of $\Lambda$ over $k$ is indecomposable if a representation $\varphi$ of $\Sigma$ over $k$ is indecomposable;
2) the representations $\gamma \otimes \varphi$ and $\gamma \otimes \varphi'$ of $\Lambda$ over $k$ are nonequivalent if representations $\varphi$ and $\varphi'$ of $\Sigma$ over $k$ are nonequivalent.

Following [2] a $k$-algebra $\Lambda$ is called wild (or of wild representation type) if it has a strict representation over $\Sigma$.

Note that the matrix $(\gamma \otimes \varphi)(\lambda)$ is obtained from the matrix $\gamma(\lambda)$ by change $x$ and $y$, respectively, on the matrices $\varphi(x)$ and $\varphi(y)$ (and $a \in k$ on the scalar matrix $aE_s$, where $E_s$ is the identity matrix of dimension $s = \deg \varphi$).

We now formulate the main result of the paper.

**Theorem.** Let $\Lambda$ be an extended Dynkin diagram of type $\tilde{D}_4$, $\tilde{E}_6$ or $\tilde{E}_7$ and $\Delta \in \mathbb{Z}^{\left|\Lambda_0\right|}$ an imaginary root of $\Lambda$. Then the algebra $Q_k(\Lambda, \Delta)$ is wild.
In proving the theorem we can obviously take $\Delta$ to be minimal positive, which we denote by $\Delta_0$.

2. Proof of the theorem for $\Lambda = \tilde{D}_4$

In this case the diagram $\Lambda$ and vector $\Delta_0$ are

![Diagram](image)

By the convention indicated above 0 denotes the strongly inner vertex, and 1, 2, 3 and 4 the outer vertices. Then the algebra $Q_k(\Lambda, \Delta_0)$, with generators $e_1, e_2, e_3, e_4$, has the relations

1') $e_i^2 = e_i$ ($1 \leq i \leq 4$);
2') $e_1 + e_2 + e_3 + e_4 = 2$.

Consider the following representation $\gamma$ of $Q_k(\Lambda, \Delta_0)$ over $\Sigma = k < x, y >$:

\[
\gamma(e_1) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
\gamma(e_2) = \begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]
Idempotent generated algebras

\[ \gamma(e_3) = \begin{pmatrix} 1 & 0 & x & y & x^2 - x + y & xy - y & 1 \\ 0 & 1 & 1 & 0 & x - 1 & y & 0 \\ 0 & 0 & 0 & 0 & -x + 1 & -y & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \]

\[ \gamma(e_4) = \begin{pmatrix} 0 & 0 & -x + 1 & -y & -x^2 + x - y & -xy + y & -1 \\ 0 & 0 & -1 & 1 & -x + 1 & -y & 0 \\ 0 & 0 & 1 & 0 & x & y & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \]

In [3] the author has proved that this representation is strict.

3. Proof of the theorem for \( \Lambda = \tilde{E}_6 \)

In this case the diagram \( \Lambda \) and vector \( \Delta_0 \) are

\[
\begin{array}{c}
\ \ \\
\ 1\ \\
\ \ \\
/ \\
/ \\
1 \quad 2 \quad 3 \quad 2 \quad 1
\end{array}
\]

We assume that the vertices 1, 3, 5 are outer, the vertices 2, 4, 6 are weakly inner (the vertex 0 is strongly inner), and the edges join the vertices 1 and 2, 3 and 4, 5 and 6, and consequently 0 with 2, 4, 6. Then the algebra \( Q_k(\Lambda, \Delta_0) \), with generators \( e_1, e_2, \ldots, e_6 \), has the relations

1’) \( e_i^2 = e_i \ (1 \leq i \leq 6); \)

2’) \( e_1e_2 = e_2e_1 = 0, e_3e_4 = e_4e_3 = 0, e_5e_6 = e_6e_5 = 0; \)

3’) \( e_1 + e_3 + e_5 + 2(e_2 + e_4 + e_6) = 3. \)

Consider the following representation \( \gamma \) of \( Q_k(\Lambda, \Delta_0) \) over \( \Sigma = k < x, y >: \)
\[
\gamma(e_1) = \begin{pmatrix}
1 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix},
\]
\[
\gamma(e_2) = \begin{pmatrix}
0 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\]
\[
\gamma(e_3) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -x & -y & 2 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\]
\[
\gamma(e_4) = \begin{pmatrix}
0 & 0 & 0 & -1 & 1 & 1 & -1 \\
0 & 0 & 0 & x & y & -x & -y \\
0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\]
\[
\gamma(e_5) = \begin{pmatrix}
0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & -x & -y & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]
We will prove that the representation $\gamma$ is strict.

Let $\varphi$ and $\varphi'$ be representations of $\Sigma$ over $k$ having the same degree: $\deg \varphi = \deg \varphi' = d$. The representation $\gamma \otimes \varphi$ (respectively, $\gamma \otimes \varphi'$) is uniquely defined by the matrices $A_s = (\gamma \otimes \varphi)(e_s)$ (respectively, $A'_s = (\gamma \otimes \varphi')(e_s)$), where $s = 1, 2, \ldots, 6$. It is natural to consider these matrices as block matrices with blocks $(A_s)_{ij}$ and $(A'_s)_{ij}$ of degree $d$ $(i, j = 1, 2, \ldots, 7)$. Then $\text{Hom}(\gamma \otimes \varphi, \gamma \otimes \varphi') = \{T \in k^{d \times d} \mid A_s T = T A'_s \text{ for each } s = 1, 2, \ldots, 6\}$.

\textbf{Lemma 1.} Let $T = (T_{ij})$, $i, j = 1, 2, \ldots, 7$, be a block matrix (over $k$) with blocks $T_{ij}$ of degree $d$, belonging to $\text{Hom}(\gamma \otimes \varphi, \gamma \otimes \varphi')$. Then $T_{ij} = 0$ if $i \neq j$ and $(i, j) \neq (1, 6), (1, 7)$, and $T_{11} = T_{22} = \ldots = T_{77}$.

\textbf{Proof.} Denote by I, II, \ldots, VI the matrix equalities $A_1 T = TA'_1$, $A_2 T = TA'_2$, \ldots, $A_6 T = TA'_6$, respectively. The (matrix) equality $(A_s T)_{ij} = (T A'_s)_{ij}$, $i, j \in \{1, 2, \ldots, 7\}$, induced by an equality $A_s T = T A'_s$, is denoted by I$(i, j)$ for $s = 1$, II$(i, j)$ for $s = 2$, \ldots, VI$(i, j)$ for $s = 6$.

It is easy to see that I(2, 1) implies $T_{21} = 0$; I(3, 1) implies $T_{31} = 0$; I(6, 4) implies $T_{64} = 0$; I(6, 5) implies $T_{65} = 0$; I(7, 4) implies $T_{74} = 0$; I(7, 5) implies $T_{75} = 0$; II(2, 4) implies $T_{24} = 0$; II(2, 5) implies $T_{25} = 0$; II(2, 6) implies $T_{26} = 0$; II(2, 7) implies $T_{27} = 0$; II(3, 4) implies $T_{34} = 0$; II(3, 5) implies $T_{35} = 0$; II(3, 6) implies $T_{36} = 0$; II(3, 7) implies $T_{37} = 0$; III(1, 2) implies $T_{12} = 0$; III(1, 3) implies $T_{13} = 0$; III(4, 2) implies $T_{42} = 0$; III(4, 3) implies $T_{43} = 0$; III(5, 2) implies $T_{52} = 0$; III(5, 3) implies $T_{53} = 0$; III(6, 2) implies $T_{62} = 0$; III(6, 3) implies $T_{63} = 0$; III(7, 2) implies $T_{72} = 0$; III(7, 3) implies $T_{73} = 0$; V(4, 6) implies $T_{46} = 0$; V(4, 7) implies $T_{47} = 0$; V(5, 6) implies $T_{56} = 0$; V(5, 7) implies $T_{57} = 0$; I(1, 4) and $T_{34} = 0$ imply $T_{14} = 0$; I(1, 5) and $T_{35} = 0$ imply $T_{15} = 0$; IV(6, 4), $T_{62} = 0$, $T_{63} = 0$ and $T_{64} = 0$ imply $T_{61} = 0$; IV(7, 4), $T_{72} = 0$, $T_{73} = 0$ and $T_{74} = 0$ imply $T_{71} = 0$; IV(4, 1) and $T_{61} = 0$ imply $T_{41} = 0$; IV(5, 1) and $T_{71} = 0$ imply $T_{51} = 0$; VI(1, 2), $T_{12} = 0$, $T_{42} = 0$, $T_{52} = 0$, $T_{62} = 0$ and $T_{72} = 0$ imply $T_{32} = 0$; V(3, 5), $T_{31} = 0$, $T_{32} = 0$ and $T_{35} = 0$ imply $T_{34} = 0$; IV(3, 7), $T_{31} = 0$, $T_{32} = 0$, $T_{35} = 0$ and $T_{47} = 0$ imply $T_{67} = 0$; IV(1, 4), VI(1, 4), $T_{12} = 0$, $T_{13} = 0$, $T_{14} = 0$, $T_{34} = 0$, $T_{64} = 0$ and $T_{74} = 0$ imply $T_{54} = 0$; IV(5, 6), $T_{51} = 0$, $T_{52} = 0$, $T_{53} = 0$, $T_{54} = 0$.
and \( T_{56} = 0 \) imply \( T_{76} = 0 \); IV(2, 5), IV(5, 7), VI(2, 7), \( T_{21} = 0 \), \( T_{25} = 0 \), \( T_{27} = 0 \), \( T_{45} = 0 \), \( T_{51} = 0 \), \( T_{52} = 0 \), \( T_{57} = 0 \), \( T_{65} = 0 \), \( T_{67} = 0 \) and \( T_{75} = 0 \) imply \( T_{23} = 0 \).

So \( T_{ij} = 0 \) when \( i \neq j \) and \((i, j) \neq (1, 6), (1, 7)\). Then it follows from IV(1, 4), IV(1, 5), IV(1, 6), IV(1, 7), III(3, 4), III(2, 4) and VI(2, 6) that \( T_{11} = T_{22} = \ldots = T_{77} \).

Therefore the representation \( \gamma \) satisfies the condition 2) (of the definition of a strict representation).

It remains to prove that \( \gamma \) satisfies the condition 1) or, in other words, \( \varphi \) is decomposable if so is \( \gamma \otimes \varphi \). We will denote by \( 0_s \) and \( E_s \) the \( s \times s \) zero and identity matrices, respectively.

Denote by \( \text{Hom}(\varphi, \varphi) \) the algebra of endomorphisms of \( \varphi \), i.e.

\[
\text{Hom}(\varphi, \varphi) = \{ S \in k^{d \times d} \mid \varphi(x)S = S\varphi(x), \varphi(y)S = S\varphi(y) \}. 
\]

Decomposability of \( \gamma \otimes \varphi \) implies that the \( k\)-algebra \( \text{Hom}(\gamma \otimes \varphi, \gamma \otimes \varphi) \) (of endomorphisms of \( \gamma \otimes \varphi \)) contains an idempotent \( T \neq 0_{7d}, E_{7d} \) (see, for example, [4, ch.V]). Then, by the lemma, the matrix \( T_0 = T_{11} = T_{22} = \ldots = T_{77} \) is an idempotent; moreover, \( T_0 \neq 0_d, E_d \), because otherwise it would follow from the equality \( T^2 = T \) that \( T = T_0 \oplus T_0 \oplus \ldots \oplus T_0 \), where \( T_0 \) occurs 7 times, or in other words \( T = 0_{7d} \) or \( T = E_{7d} \), respectively. Since \( T_0 \) belong to the algebra \( \text{Hom}(\varphi, \varphi) = \{ S \in k^{d \times d} \mid \varphi(x)S = S\varphi(x), \varphi(y)S = S\varphi(y) \} \) (see the condition b)), the representation \( \varphi \) is decomposable (see again [4, ch.V]).

4. **Proof of the theorem for** \( \Lambda = \tilde{E}_7 \)

In this case the diagram \( \Lambda \) and vector \( \Delta_0 \) are

\[
\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 3 & 2 & 1 \\
\end{array}
\]

We assume that the vertices 1, 4, 7 are outer, the vertices 2, 3, 5, 6 are weakly inner (the vertex 0 is strongly inner), and the edges join the
vertices 1 and 2, 2 and 3, 4 and 5, 5 and 6, and consequently 0 with 3, 6, 7. Then the algebra $Q_k(\Lambda, \Delta_0)$, with generators $e_1, e_2, \ldots, e_7$, has the relations

1') $e_i^2 = e_i$ ($1 \leq i \leq 7$);

2') $e_1e_2 = e_2e_1 = 0, e_2e_3 = e_3e_2 = 0, e_1e_3 = e_3e_1 = 0, e_4e_5 = e_5e_4 = 0, e_5e_6 = e_6e_5 = 0, e_4e_6 = e_6e_4 = 0$;

3') $e_1 + e_4 + 2(e_2 + e_5 + e_7) + 3(e_3 + e_6) = 4$.

Consider the following representation $\gamma$ of $Q_k(\Lambda, \Delta_0)$ over $\Sigma = k < x, y >$:

$$
\gamma(e_1) = \begin{pmatrix}
0 & 0 & 1 & 0 & -3 & 0 & 0 & 3 & 0 \\
0 & 1 & 0 & -3 & 0 & 0 & x & y \\
0 & 0 & 1 & 0 & -3 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
$$

$$
\gamma(e_2) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 & -3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 0 & -3 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
$$

$$
\gamma(e_3) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 3 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
$$
\[ \gamma(e_4) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 3 & 3 & 9 & 9 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -3 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \]

\[ \gamma(e_5) = \begin{pmatrix}
1 & 0 & 1 & 0 & 0 & -3 & -3 & -12 & -9 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & x & y \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \]

\[ \gamma(e_6) = \begin{pmatrix}
0 & 0 & -1 & 0 & 1 & 0 & 1 & 3 & 3 \\
1 & 0 & -1 & 0 & -1 & 0 & -x & -3 & -y \\
0 & 1 & 0 & -1 & 0 & -1 & -3 & -3 & -3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \]

\[ \gamma(e_7) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. \]
We will prove that the representation $\gamma$ is strict.

Let $\varphi$ and $\varphi'$ be representations of $\Sigma$ over $k$ having the same degree: $\deg \varphi = \deg \varphi' = d$. The representation $\gamma \otimes \varphi$ (respectively, $\gamma \otimes \varphi'$) is uniquely defined by the matrices $A_s = (\gamma \otimes \varphi)(e_s)$ (respectively, $A'_s = (\gamma \otimes \varphi')(e_s)$), where $s = 1, 2, \ldots, 7$. It is natural to consider these matrices as block matrices with blocks $(A_s)_{ij}$ and $(A'_s)_{ij}$ of degree $d$ $(i, j = 1, 2, \ldots, 9)$. Then $\Hom(\gamma \otimes \varphi, \gamma \otimes \varphi') = \{ T \in k^{9d \times 9d} | A_s T = T A'_s \text{ for each } s = 1, 2, \ldots, 7 \}$.

**Lemma 2.** Let $T = (T_{ij})$, $i, j = 1, 2, \ldots, 9$, be a block matrix (over $k$) with blocks $T_{ij}$ of degree $d$, belonging to $\Hom(\gamma \otimes \varphi, \gamma \otimes \varphi')$. Then $T_{ij} = 0$ if $i \neq j$ and $(i, j) \neq (1, 8), (1, 9)$, and $T_{11} = T_{22} = \ldots = T_{99}$.

**Proof.** Denote by I, II, \ldots, VII the matrix equalities $A_1 T = TA'_1$, $A_2 T = TA'_2$, \ldots, $A_7 T = TA'_7$, respectively. The (matrix) equality $(A_s T)_{ij} = (TA'_s)_{ij}$, $i, j \in \{1, 2, \ldots, 9\}$, induced by an equality $A_s T = TA'_s$, is denoted by $I(i, j)$ for $s = 1$, II($i, j$) for $s = 2$, \ldots, VII($i, j$) for $s = 7$.

It is easy to see that VII($i, j$) implies $T_{ij} = 0$ for each $(i, j) \in \{1, 4, 5, 8, 9\} \times \{2, 3, 6, 7\}$ and each $(i, j) \in \{2, 3, 6, 7\} \times \{1, 4, 5, 8, 9\}$; II(1, 4) and $T_{12} = 0$ imply $T_{14} = 0$; II(1, 5) and $T_{13} = 0$ imply $T_{15} = 0$; II(4, 1) and $T_{51} = 0$ imply $T_{41} = 0$; II(4, 8) and $T_{68} = 0$ imply $T_{48} = 0$; II(4, 9) and $T_{69} = 0$ imply $T_{49} = 0$; II(5, 1) and $T_{71} = 0$ imply $T_{51} = 0$; II(5, 8) and $T_{78} = 0$ imply $T_{58} = 0$ II(5, 9) and $T_{79} = 0$ imply $T_{59} = 0$; II(8, 4) and $T_{82} = 0$ imply $T_{84} = 0$; II(8, 5) and $T_{83} = 0$ imply $T_{85} = 0$; II(9, 4) and $T_{92} = 0$ imply $T_{94} = 0$; II(9, 5) and $T_{93} = 0$ imply $T_{95} = 0$; III(6, 2) and $T_{82} = 0$ imply $T_{62} = 0$; III(6, 3) and $T_{83} = 0$ imply $T_{63} = 0$; III(7, 2) and $T_{92} = 0$ imply $T_{72} = 0$; III(7, 3) and $T_{93} = 0$ imply $T_{73} = 0$; III(6, 1) and $T_{61} = 0$ imply $T_{81} = 0$; III(1, 9), $T_{13} = 0$, $T_{15} = 0$, $T_{17} = 0$ and $T_{69} = 0$ imply $T_{89} = 0$; III(6, 9), $T_{63} = 0$, $T_{65} = 0$, $T_{69} = 0$ and $T_{89} = 0$ imply $T_{67} = 0$; III(7, 1) and $T_{71} = 0$ imply $T_{91} = 0$; IV(2, 6), $T_{21} = 0$ and $T_{24} = 0$ imply $T_{26} = 0$; IV(2, 7), $T_{21} = 0$ and $T_{25} = 0$ imply $T_{27} = 0$; IV(3, 6), $T_{31} = 0$ and $T_{34} = 0$ imply $T_{36} = 0$; IV(3, 7), $T_{31} = 0$ and $T_{35} = 0$ imply $T_{37} = 0$; VI(1, 2), $T_{12} = 0$, $T_{52} = 0$, $T_{72} = 0$, $T_{92} = 0$ and $T_{99} = 0$ imply $T_{32} = 0$; VI(2, 7), $T_{21} = 0$, $T_{27} = 0$, $T_{47} = 0$, $T_{67} = 0$, $T_{87} = 0$ and $T_{97} = 0$ imply $T_{23} = 0$; VI(2, 5), $T_{21} = 0$, $T_{23} = 0$, $T_{25} = 0$, $T_{65} = 0$, $T_{85} = 0$ and $T_{95} = 0$ imply $T_{45} = 0$; VI(1, 4), $T_{12} = 0$, $T_{34} = 0$, $T_{74} = 0$, $T_{84} = 0$ and $T_{84} = 0$ imply $T_{54} = 0$; II(5, 6), $T_{52} = 0$, $T_{54} = 0$ and $T_{56} = 0$ imply $T_{76} = 0$; III(5, 8), $T_{51} = 0$, $T_{52} = 0$, $T_{54} = 0$, $T_{56} = 0$ and $T_{78} = 0$ imply $T_{98} = 0$.

So $T_{ij} = 0$ when $i \neq j$ and $(i, j) \neq (1, 8), (1, 9)$. Then it follows from III(1, 6), III(1, 8), III(5, 7), III(5, 9), VI(1, 3), VI(1, 5) VI(2, 4) and VI(2, 6). that $T_{11} = T_{22} = \ldots = T_{99}$. $\square$
The final part of the proof is analogous to that in the case $\Lambda = \tilde{E}_6$ (see Section 3).

References


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