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Groups, in which almost all subgroups are near to normal

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Abstract. A subgroup H of a group G is said to be nearly normal, if H has a finite index in its normal closure. These subgroups have been introduced by B.H. Neumann. In a present paper is studied the groups whose non polycyclic by finite subgroups are nearly normal. It is not hard to show that under some natural restrictions these groups either have a finite derived subgroup or belong to the class S_1F (the class of soluble by finite minimax groups). More precisely, this paper is dedicated of the study of S_1F groups whose non polycyclic by finite subgroups are nearly normal.

Let ν be a subgroup-theoretical property. This property can be internal to the group as, for example, in the cases when ν denotes the property of being a normal subgroup, subnormal subgroup, almost normal subgroup, or permutable subgroup, and external to the groups, as in the case when ν refers to belonging to some class of groups $\mathfrak{X}$. If G is a group then let

$$\mathcal{S}_{\nu}(G) = \{H | H \text{ is a subgroup of } G \text{ and } H \text{ has a property } \nu\},$$

$$\mathcal{S}_{non-\nu}(G) = \{H | H \text{ is a subgroup of } G \text{ and } H \text{ does not have a property } \nu\}.$$

One of the important problem of Group Theory is the study of influence of the families $\mathcal{S}_{\nu}(G)$ and $\mathcal{S}_{non-\nu}(G)$ on a structure of group for the most natural properties ν. The starting point of such researches is the paper of R. Dedekind [1], in which finite groups with all subgroups are normal have been described; that is the family $\mathcal{S}_{non-norm}(G)$ is empty. The next important paper in this direction was the work of G.A. Miller and H. Moreno [2], where finite groups, in which every proper subgroup is abelian, are considered; that is $\mathcal{S}_{non-ab}(G) = \{G\}$. The paper of O.Yu. Schmidt [2], where have been described the finite groups with all proper subgroups are nilpotent ($\mathcal{S}_{non-nil}(G) = \{G\}$), has played also an significant role in this series. These researches inspired the systematic study of groups, in which the family $\mathcal{S}_{non-\nu}(G)$ is "very small" or $\mathcal{S}_{\nu}(G)$ is "very large" in some sense. This direction became very fruitful. Many prominent mathematicians published significant amount of articles and monographs dedicated to this problematic. In the current paper we will consider groups with the restrictions on the family $\mathcal{S}_{non-nn}(G)$ of all non nearly normal subgroups. A subgroup H of a group G is said to be nearly normal (in G), if H has a finite index in its normal closure H^G. Such subgroups without any special title have been introduced and considered first by B.H. Neumann [4]. In [5] they have been named finitely-normal. We think that this is not a best possible name. In [6] these subgroup have been named more successfully as near normal subgroup. Lately the interest to such subgroups increased significantly (see, for example, [7, 8]). In the paper [4] B.H. Neumann characterized groups, all subgroups of which are nearly normal (that is the family $\mathcal{S}_{non-nn}(G)$ is empty) as the groups with finite derived subgroup. The subsequent study of groups, in which the family $\mathcal{S}_{non-nn}(G)$ is “significantly small”, has been continued in the papers of other authors (see, for example, [5, 7, 8]). In our paper we consider the groups, in which the family $\mathcal{S}_{non-nn}(G)$ consists of only polycyclic-by-finite subgroups. In this connection note that the groups, in which the family of all non-normal subgroups consists of polycyclic-by-finite subgroups, have been considered by L.A. Kurdachenko, V.V. Pylaev and G. Cutolo (see [9, 10, 11]); the groups, in which all non almost normal subgroups are polycyclic-by-finite, have been studied by S. Franciosi, F. de Giovanni and L.A. Kurdachenko [12] (more precisely, this paper dedicated to the groups, whose non-finitely generated subgroups are almost normal and the description of groups, whose not polycyclic-by-finite subgroups are almost normal, could be derived from these results). As the series of examples constructed by A.Yu. Ol'shanskij [12] show the groups, whose proper subgroups are polycyclic-by-finite have very complicated structure and their description is not achievable now. Therefore the study of groups, whose non-(polycyclic-by-finite) subgroups are nearly normal, it is expedient to carry out by some additional conditions, in some way similar to generalized solubility. It is not hard to show that under such conditions these groups either have a finite derived subgroup or belong to the class $\mathfrak{S}_1\mathfrak{F}$. Hence the case of $\mathfrak{S}_1\mathfrak{F}$-groups is the basic here. This paper is dedicated these case.

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Note first some elementary properties of the nearly normal subgroups. Let H, L be the nearly normal subgroups of a group G, then $K =< H, L >$ and $H \cap L$ are nearly normal. Indeed, we have $H^G = x_1 H \cup \ldots \cup x_n H$, $L^G = Ly_1 \cup \ldots \cup Ly_m$. If $u \in H^G$, $\nu \in L^G$, then $u \in x_j H$, $\nu \in Ly_t$ for some $1 \leq j \leq n$, $1 \leq t \leq m$, so that $u\nu \in x_j HLy_t$. Hence $H^G L^G = \bigcup_{1\leq j\leq n,\ 1\leq t\leq m} x_j HLy_t$. Furthermore $HL \leq K$ implies $x_j HLy_t \leq x_j Ky_t = x_j y_t(y_t^{-1}Ky_t)$, therefore

$$H^G L^G = \bigcup_{1\leq j\leq n,\ 1\leq t\leq m} x_j y_t(y_t^{-1}Ky_t).$$

Using 4.1 of [14] we obtain that at least one of the subgroups $y_t^{-1}Ky_t$ has a finite index in $H^G L^G$, and hence K has a finite index in $H^G L^G$. An obviously inclusion $K^G \leq H^G L^G$ implies that K is nearly normal in G.

Consider now an intersection $H \cap L$. We have

$$/H^G \cap L^G/ : H \cap L/ = /H^G \cap L^G : H^G \cap L^G \cap L//H^G \cap L^G \cap L : H \cap L/.$$

Using now the properties of indexes, we obtain that

$$/H^G \cap L^G : H^G \cap L^G \cap L/ = /(H^G \cap L^G)L : L/ \leq /L^G : L/ \text{ and}$$

$$/H^G \cap L^G \cap L : H \cap L/ = /(H^G \cap L)H : H/ \leq /H^G : H/,$$

what implies that $/H^G \cap L^G : H \cap L/$ is finite. An obviously inclusion $(H \cap L)^G \leq H^G \cap L^G$ implies that $H \cap L$ is nearly normal in G. We will use often these properties without special references.

Lemma 1. Let G be a group, whose non polycyclic-by-finite subgroups are nearly normal. If H is a normal subgroup of G such that every subgroup including H is non polycyclic-by-finite, then G/H has finite derived subgroup.

In fact, every subgroup including H is nearly normal. In this case $[G/H, G/H]$ is finite by a result due to B. Neumann [4].

A group G is called an $\mathfrak{F}$-perfect if it does not include the proper subgroups of finite index. The $\mathfrak{F}$-perfect groups are the antipode (in some sense) of the residually finite groups. In every group G the subgroup generated by all its $\mathfrak{F}$-perfect subgroups is likewise $\mathfrak{F}$-perfect. This subgroup is called the $\mathfrak{F}$-perfect part of a group G.

Lemma 2. Let G be a group and H be an $\mathfrak{F}$-perfect subgroup of G. If H is nearly normal, then it is normal.

Indeed, since the index $/H^G : H/$ is finite and H is $\mathfrak{F}$-perfect, then H coincides with the $\mathfrak{F}$-perfect part of H^G. The $\mathfrak{F}$-perfect part is a characteristic subgroup, in particular, it is normal in G.

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Theorem 1. Let $G \in \mathfrak{S}_1\mathfrak{F}$. Every non polycyclic-by-finite subgroup of G is nearly normal if and only if G is the groups of one of the following types:

(1) G is a group with finite derived subgroup;

(2) G is an almost Prüfer group.

Proof. Suppose, that G has an infinite derived subgroup. Being periodic G is a Chernikov group. Let D be a divisible ($\mathfrak{F}$-perfect) part of G. Assume that D is not a Prüfer group. Let K be a Prüfer subgroup of D. We have $D = K \times B$ where B is a divisible subgroup, in particular, it is $\mathfrak{F}$-perfect. By Lemma 2 the both subgroups K and B are normal in G. Lemma 1 yields that $[G/K, G/K]$ and $[G/B, G/B]$ are finite. Since $K \cap B =< 1 >$, then by Remak's theorem $G \hookrightarrow G/K \times G/B$. In turn out this embedding proves that $[G, G]$ is finite. This contradiction proves that D a Prüfer subgroup, that is G has a type (2).

Conversely, let G is a group of types (1) – (2). If G has a type (1), then every its subgroup is nearly normal. Consider now the case, when G is a group of a type (2). Let H be a subgroup of G and suppose, that H is not polycyclic-by-finite. Denote again by D a divisible ($\mathfrak{F}$-perfect) part of G. Since G/D is finite, an intersection $H \cap D$ is infinite. Then $H \cap D = D$, that is $D \leq H$ and hence H has a finite index in G. Clearly then H is nearly normal in G. □

A further study of a non-periodic case splits on two following situations:

(1) a group includes the infinite periodic subgroups; and

(2) all periodic subgroups are finite.

Consider these situation in this turn.

Lemma 3. Let $G \in \mathfrak{S}_1\mathfrak{F}$. Suppose that G is not periodic, $[G, G]$ is infinite and $\zeta(G)$ does not include the Prüfer subgroups. If every non polycyclic-by-finite subgroup of G is nearly normal, then G includes a normal Prüfer subgroup D such that G/D is a finitely generated group with finite derived subgroup.

Proof. Let P be an infinite periodic subgroup of G. Choose a maximal periodic subgroup T including P. Since Let $G \in \mathfrak{S}_1\mathfrak{F}$, T is Chernikov. Let D be a divisible ($\mathfrak{F}$-perfect) part of T. Assume that D is not a Prüfer group. Let K be a Prüfer subgroup of D. We have $D = K \times B$ where B is a divisible subgroup, in particular, it is $\mathfrak{F}$-perfect. By Lemma 2

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the both subgroups K and B are normal in G. Lemma 1 yields that $[G/K, G/K]$ and $[G/B, G/B]$ are finite. Since $K \cap B =< 1 >$, then by Remak's theorem $G \hookrightarrow G/K \times G/B$. In turn out this embedding proves that $[G, G]$ is finite. This contradiction proves that D a Prüfer subgroup. A factor-group G/D has a finite derived subgroup, in particular, it is an FC-group. Since $\zeta(G)$ does not include D, there is an element $g \in G$ such that $[g, D] \neq < 1 >$. If we suppose that $[g, D] \neq D$, then $[g, D]$ is finite. It follows that a derived subgroup of $< D, g >$ is finite, in particular, $< D, g >$ is an FC-group. However the center of an FC-group includes every divisible subgroup. This contradiction proves an equation $[g, D] = D$. Put $C/D = C_{G/D}(gD)$ and let $x \in C$. Then $g^x = gy$ for some element $y \in D$. By $D = [g, D]$ we have $y = [g, u]$ for some element $u \in D$. Therefore $g^x = gy = g[g, u] = g^u$, and this implies $xu^{-1} \in C_G(g)$, which proves an equation $C = DC_G(g)$. By the selection of g we have $D \cap C_G(g) \neq D$, and hence $D \cap C_G(g)$ is finite. Let $E = C_G(g)$ and suppose that E is not polycyclic-by-finite. Then E is nearly normal, i.e. $/E^G : E/$ is finite. A normal closure E^G is not polycyclic-by-finite, moreover, every subgroup including E^G also is not polycyclic-by-finite and hence is nearly normal. In other words, every subgroup of G/E^G is nearly normal. Then G/E^G has a finite derived subgroup by a result due to B. Neumann [4]. Since $/E^G : E/$ is finite, $F = D \cap E^G$ is finite. By Remak's theorem $G/F \hookrightarrow G/D \times G/E^G$, what proves that $[G/F, G/F]$ is finite. Together with a finiteness of F it follows that $[G, G]$ is finite. This contradiction proves that E is polycyclic-by-finite. Since G/D is an FC-group, C has a finite index in G. From $C = DE$ we obtain that C/D is finitely generated, and hence so is G/D. □

Lemma 4. Let G be a group and A be an abelian torsion-free subgroup having finite 0-rank. If A is nearly normal, then A includes a G-invariant abelian subgroup B such that A^G/B is finite.

Proof. Since A is nearly normal, $/A^G : A/$ is finite. There exists a positive integer n such that $B = (A^G)^n \leq A$. Clearly B is G-invariant and a factor $A^{G/}B$ is bounded. The finiteness of $/A^G : A/$ implies that A^G has a finite special rank. But a bounded abelian-by-finite group of finite special rank is finite. Thus A^G/B is finite. □

Lemma 5. Let G be a non-periodic locally (soluble-by-finite) group. Suppose also that $[G, G]$ is infinite and $\zeta(G)$ includes the Prüfer subgroups. If all non-polycyclic-by-finite subgroups of G are nearly normal, then $[G/D, G/D]$ is finite and a section $A/(A \cap D)$ is finitely generated for every abelian subgroup A.

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Proof. Suppose the contrary. Let $A/(A \cap D)$ is not finitely generated. Since $D \leq \zeta(G)$, AD is abelian. In other words, we can assume that $D \leq A$. By the properties of the divisible subgroups of abelian groups (see, for example, [15, Theorem 21.2]) $A = D \times E$ for some subgroup E. By our assumption on A/D a subgroup E can not be finitely generated. Then it is nearly normal. Lemma 4 yields that E includes a G-invariant torsion-free subgroup B of finite index. In particular, B is not finitely generated. By Lemma 1 the both factor-groups G/D and G/B have the finite derived subgroups. By $D \cap B \leq D \cap E =< 1 >$ we obtain an embedding $G \hookrightarrow G/D \times G/B$, which proves that $[G, G]$ is finite. This contradiction shows that $A/(A \cap D)$ is finitely generated. □

Corollary 1. Let $G \in \mathfrak{S}_1\mathfrak{F}$. Suppose that G is not periodic, $[G, G]$ is infinite and $\zeta(G)$ includes a Prüfer subgroup D. If every non polycyclic-by-finite subgroup of G is nearly normal and G/D is not finitely generated, then every abelian subgroup of G has an infinite index in G.

Proof. Since D is not polycyclic-by-finite, $[G/D, G/D]$ is finite. Suppose that G includes an abelian normal subgroup A of finite index. Being $\mathfrak{F}$-perfect D lies in A. If A/D is finitely generated, then so is and G/D. This contradiction shows that A/D can not be finitely generated. However this contradicts to Lemma 5. □

Corollary 2. Let $G \in \mathfrak{S}_1\mathfrak{F}$. Suppose that G is not periodic, $[G, G]$ is infinite and $\zeta(G)$ includes a Prüfer subgroup D. If all non polycyclic-by-finite subgroups of G are nearly normal, then a section $A/(A \cap D)$ is finitely generated for every FC-subgroup A.

Proof. Suppose the contrary, let $A/(A \cap D)$ is not finitely generated. Since $D \leq \zeta(G)$, AD is an FC-subgroup. In other words, we can assume that $D \leq A$. By Lemma 1 $[G/D, G/D]$ is finite. It follows that the set T of all elements having finite orders is a (characteristic) subgroup of G. Moreover, G/T is abelian. Since Let $G \in \mathfrak{S}_1\mathfrak{F}$, T is Chernikov. Lemma 5 proves that T/D is finite. Put $C/D = C_{A/D}(T/D)$, then A/C is finite. Since $(C/D) \cap (T/D) \leq \zeta(C/D)$ and $(C/D)/((C/D) \cap (T/D)) \cong (C/D)(T/D)/(T/D) \cong CT/T$ is abelian, then C/D is nilpotent. Finally, an inclusion $D \leq \zeta(G)$ implies that C is nilpotent. A factor-group C/D is not finitely generated, because A/C is finite. Put $Z = \zeta(C)$, then C/Z is periodic (see, for example, [16, Theorem 4.32]). Since Z has a finite special rank, it includes a finitely generated subgroup K such that C/K is periodic. By the nilpotency of C/K we have $C/K = P/K \times Q/K$ where P/K is a Sylow p-subgroup, $p = \Pi(D)$, Q/K is a Sylow p'-subgroup. Since C/D is infinite, either DK/K has an infinite index in

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P/K or Q/K is infinite. In every case C/K includes a normal; infinite subgroup U/K such that $U/K \cap DK/K =< 1 >$. Then $U \cap D = L$ is a finite subgroup. Clearly U is not polycyclic, and Lemma 1 shows that $[C/U, C/U]$ is finite. Using again Remak's theorem we obtain an embedding $C/L \hookrightarrow E/D \times E/U$, which proves that $[C/L, C/L]$ is finite. Then by finiteness of L a derived subgroup $[C, C]$ is finite. In this case $G/[C, C]$ is abelian-by-finite, what contradicts to Corollary 1 of Lemma 5. This contradiction shows that $A/(A \cap D)$ is finitely generated. □

Corollary 3. Let $G \in \mathfrak{S}_1\mathfrak{F}$. Suppose that G is not periodic, $[G, G]$ is infinite and $\zeta(G)$ includes a Prüfer subgroup D. If every non polycyclic-by-finite subgroup of G is nearly normal and G/D is not finitely generated, then every FC-subgroup of G has an infinite index in G. In particular, $G/FC(G)$ is infinite.

Proof. By Lemma 1 $[G/D, G/D]$ is finite. Suppose that G includes a normal FC-subgroup A of finite index. Being $\mathfrak{F}$-perfect D lies in A. If we suppose that A/D is finitely generated, then so is and G/D. This contradiction shows that A/D can not be finitely generated. But this contradicts to Corollary 2 of Lemma 5. □

Lemma 6. Let $G \in \mathfrak{S}_1\mathfrak{F}$. Suppose that G is not periodic, $[G, G]$ is infinite and $\zeta(G)$ includes a Prüfer subgroup D. If every non polycyclic-by-finite subgroup of G is nearly normal, then

(i) $[G/D, G/D]$ is finite;

(ii) If H is a not finitely generated subgroup of G, then $D \leq H$.

Proof. By Lemma 1 $[G/D, G/D]$ is finite. Suppose that H does not include D. This means, that an intersection $H \leq D$ is finite. Since $G \in \mathfrak{S}_1\mathfrak{F}$, H includes an abelian normal torsion-free subgroup A such that H/A is finite [17, Lemma 3]. Clearly A is not finitely generated. But this contradicts to Lemma 5. □

Let G be an abelian minimax group. Choose in G a finitely generated torsion-free subgroup H such that G/H is periodic (and hence Chernikov). Let D/H be a divisible part of G/H. Put $\mathbf{Sp}(G) = \Pi(D/H)$. If K is another finitely generated torsion-free subgroup of G such that G/K is periodic, then the both factors $H/(H \cap K)$ and $K/(H \cap K)$ are finite. It follows that the divisible parts of G/H and G/K are isomorphic. In other words, a set $\mathbf{Sp}(G)$ is an invariant of a group G. Let $p \notin \Pi(G/H)$, then H/H^p is a finite Sylow p-subgroup of G/H^p and $G/H^p = H/H^p \times R/L^p$ (see, for example, [15, Theorem 27.5]). This shows that $G \neq G^p$.

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Lemma 7. Let $G \in \mathfrak{S}_1\mathfrak{F}$. Suppose that G is not periodic, $[G, G]$ is infinite and $\zeta(G)$ includes a Prüfer subgroup D. If every non polycyclic-by-finite subgroup of G is nearly normal, then

(i) $[G/D, G/D]$ is finite;

(ii) If H either G/D is finitely generated or G includes a normal periodic subgroup T such that T/D is finite and G/T is an abelian minimax torsion-free group such that $\mathbf{Sp}(G/T) = p$, where $p = \Pi(D)$.

Proof. By Lemma 1 $[G/D, G/D]$ is finite. It follows that a set T of all elements having finite orders is a subgroup. Since $G \in \mathfrak{S}_1\mathfrak{F}$, T is Chernikov. By Lemma 5 T/D is finite. Suppose that G/T is not finitely generated. A group G includes a normal subgroup $H \geq D$ such that H/D is an abelian torsion-free group and G/H is finite [17, Lemma 3]. In particular, H is a nilpotent of class 2 group. For each $h \in H$ the mapping $\Theta_h : x \longrightarrow [h, x]$, $x \in H$, is an endomorphism, so that $[H, h] = \mathbf{Im}\Theta_h \cong H/\mathbf{Ker}\Theta_h = H/C_H(h)$. Since $[H, h] \leq D$, then either $H/C_H(h)$ is a finite cyclic p-group or $H/C_H(h)$ is a Prüfer p-subgroup where $p = \Pi(D)$. Since $G \in \mathfrak{S}_1\mathfrak{F}$, an abelian torsion-free subgroup H/D has a finite 0-rank. Let $h_1D, \ldots, h_rD$ be a maximal $\mathbb{Z}$ independent set of H/D and put $V = C_H(h_1, \ldots, h_r)$. An obvious equation $C_H(h_1, \ldots, h_r) = C_H(h_1) \cap \ldots \cap C_H(h_r)$ and Remak's theorem give an embedding $H/V \hookrightarrow H/C_H(h_1) \times \ldots \times C_H(h_r)$, which proves that H/V is a Chernikov p-group. Suppose that VD/D is not finitely generated. A subgroup $W =< h_1, \ldots, h_r >$ lies in the center of V, in particular, it is normal. By the selection of W a factor V/W is periodic. Since V is nilpotent, $V/W = \mathbf{Dr}_{q \in \Pi(V/W)} S_q/W$ where S_q/W is a Sylow q-subgroup of V/W. Since $V \in \mathfrak{S}_1\mathfrak{F}$, the Sylow subgroups of its periodic factor-group V/W are Chernikov. Remark that the center of a periodic nilpotent group includes every divisible subgroup (see, for example, [16, Lemma 3.13]), then every S_q/W is central-by -finite, in particular, it is an FC-group. Then V/W is a periodic FC-group. Since W is finitely generated, $W \cap D$ is finite. An embedding $V/(W \cap D) \hookrightarrow V/W \times V/D$ and the fact that V/D is abelian prove that $V/(W \cap D)$ is an FC-group. It follows that V is an FC-group because $W \cap D$ is finite. By Corollary 3 of Lemma 5 V/D is finitely generated. Hence H/V is infinite. Then there is an index j, $1 \leq j \leq r$, such that $H/C_H(h_j)$ is a Prüfer p-group. We have already noted above that H/V is a Chernikov p-group. Since V/D is finitely generated, H/D is an abelian minimax torsion-free group such that $\mathbf{Sp}(H/D) = p$. Finally, a finiteness of G/H implies that G/T is an abelian minimax torsion-free group such that $\mathbf{Sp}(G/T) = p$. □

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Now we may collect the all above results and to finish a consideration of a case when a group includes the infinite periodic subgroups.

Theorem 2. Let $G \in \mathfrak{S}_1\mathfrak{F}$. Suppose that G is not periodic and includes an infinite periodic subgroup. Every non polycyclic-by-finite subgroup of G is nearly normal if and only if G is the groups of one of the following types:

(1) G has a finite derived subgroup.

(2) $[G,G]$ includes a Prüfer subgroup D such that D is normal in G, $[G,G]/D$ is finite and G/D is finitely generated.

(3) G satisfies the following conditions:

(3A) $[G,G]$ includes a Prüfer subgroup D such that $D \leq \zeta(G)$;

(3B) the set of all elements of G/D having finite orders is a finite subgroup T/D;

(3C) G/T is an abelian minimax group;

(3D) $\mathbf{Sp}(G/T) = \{p\} = \Pi(D)$ where p is a prime;

(3E) if A is an FC-subgroup, then $A/(A \cap D)$ is finitely generated. In particular, every non finitely generated subgroup of G includes D.

Proof. Assume that $[G,G]$ is infinite. If $\zeta(G)$ does not include the Prüfer subgroups, then by Lemma 3 G is a group of type (2). Suppose now that $\zeta(G)$ includes a Prüfer subgroup D. By Lemma 1 $[G/D, G/D]$ is finite. It follows that a set T of all elements having finite orders is a subgroup. Since $G \in \mathfrak{S}_1\mathfrak{F}$, T is Chernikov. By Lemma 5 T/D is finite. If G/D is finitely generated, then again G is a group of type (2). Suppose that G/D is not finitely generated. By Lemma 7 G satisfies (3C), (3D). Finally, a condition (3E) follows from Corollary 2 of Lemma 5 and Lemma 6.

Conversely, let G be a group of types (1) — (3). If G is a group of type (1), then each its subgroup is nearly normal. Let H be a subgroup of G and suppose that H is not polycyclic-by-finite. First consider a case when G is a group of type (2). Since G/D is finitely generated, $H \cap D$ can not be finite. Then $H \cap D = D$, that is $D \leq H$. But every subgroup of G/D is nearly normal, so that H is nearly normal in G. Finally, let G be a group of type (3). Suppose that H does not include D. Then $H \cap D$ is finite. The conditions (3B), (3C) and $H/(H \cap D) \cong (HD)/D$ together with finiteness of $[G/D, G/D]$ imply that $[H,H]$ is finite. In particular, H is an FC-subgroup. Since $H \cap D$ is finite, $H/(H \cap D)$ is not finitely

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generated, what contradicts (3E). This contradiction proves an inclusion $D \leq H$. Now we remark again that every subgroup of G/D is nearly normal, so that H is nearly normal in G. □

The next result shows that a study of our groups is possible up to the factor-group by finite normal subgroups.

Proposition 1. Let G be a group and F be a normal finite subgroup of G. A subgroup H is nearly normal in G if and only if HF/F is nearly normal in G/F.

Proof. If H is nearly normal in G, then $/H^G : H/$ is finite. Since F is finite, $/H^G F : H^G/$ is finite. It implies that $/H^G F : H/$ is finite. In particular, the index $/H^G F : HF/$ is finite, so that $/(HF)^G : HF/$ is also finite. It proves that HF is nearly normal in G, therefore HF/F is nearly normal in G/F. Conversely, let HF/F is nearly normal in G/F. Then HF is nearly normal in G. Since $/HF : H/$ is finite, a subgroup H has a finite index in some normal subgroup. In particular, the index $/H^G : H/$ is finite, so that H is nearly normal in G. □

The next finally phase is a consideration of a case when all periodic subgroups are finite. Proposition 1 shows that for the description of the locally nilpotent groups with finite periodic part it is sufficient to describe only torsion- free locally nilpotent groups. A description of such groups splits on two situation depending of a derived subgroup.

Lemma 8. Let G be a locally nilpotent torsion-free group of finite 0-rank. Suppose that G is not finitely generated. If every non polycyclic subgroup of G is nearly normal, then $\zeta(G)$ is not finitely generated. In particular, G is nilpotent of class at most 2.

Proof. Suppose that $[G, G]$ is infinite. By Maltsev's theorem (see, for example, [18, Theorem 6.36]) G is nilpotent. Since G is not finitely generated, its center Z can not be finitely generated [19, Lemma 2.6]. By Lemma 1 $[G/Z, G/Z]$ is finite. On the other hand, G/Z is torsion-free (see, for example, [16, Theorem 2.25]). This means that G/Z is abelian. □

Corollary. Let G be a non-abelian locally nilpotent torsion-free group of finite 0-rank. Suppose that G is not finitely generated but $[G, G]$ is finitely generated. If every non polycyclic subgroup of G is nearly normal, then $G/\zeta(G)$ is abelian and finitely generated.

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Proof. By Lemma 8 $\zeta(G)$ is not finitely generated and $G/\zeta(G)$ is abelian. For every element $g \in G$ the mapping $\phi_g : x \longrightarrow [x, g], x \in G$, is an endomorphism, so that

$$[G, g] = \mathbf{Im}\phi_g \cong A/\mathbf{Ker}\phi_g = G/C_G(g).$$

Since $[G, G]$ is finitely generated, $[G, g]$ is likewise finitely generated. Then $G/C_G(g)$ is finitely generated for each $g \in G$. Let $\{g_1\zeta(G), \dots, g_t\zeta(G)\}$ be a maximal $\mathbb{Z}$-independent set of $G/\zeta(G)$, $H =< g_1, \dots, g_t >$. By Remak's theorem

$$G/C_G(H) \hookrightarrow G/C_G(g_1) \times \ldots \times G/C_G(g_t),$$

which proves that $G/C_G(H)$ is finitely generated. Put $D = H\zeta(G)$, then $C_G(H) = C_G(D)$, in particular, $G/C_G(D)$ is finitely generated. By the selection of D for every element $g \in G$ there is an integer m such that $g^m \in D$. For each element $x \in C_{G(}D)$ we have $g^m x = x g^m$. But in nilpotent torsion-free group G a last equation implies $gx = xg$, which shows that $C_G(D) \leq \zeta(G)$, that is $C_G(D) = \zeta(G)$. In turn out it implies that a factor-group $G/\zeta(G)$ is finitely generated. □

Let A be an abelian torsion-free group. Denote by $\mathbf{PM}_{fg}(A)$ the intersection of all pure not finitely generated subgroups of A. A subgroup $\mathbf{PM}_{fg}(A)$ is called a pure polycyclic monolith of A.

A subgroup $\mathbf{PM}_{fg}(A)$ has been introduced in a paper [12]. We remark that it [12] has been used the other name and other designation.

Lemma 9. Let A be an abelian torsion-free group. If $< 1 > \neq \mathbf{PM}_{fg}(A)$, then $r_0(A)$ is finite and A includes a pure finitely generated subgroup B such that $r_0(A/B) = 1$.

Proof. Put $C = \mathbf{PM}_{fg}(A)$. Then every proper pure subgroup of C is finitely generated. Let M be a maximal $\mathbb{Z}$-independent set of C. We may extend M to a maximal $\mathbb{Z}$-independent set L of a group A. For an element $a \in M$ choose in A a maximal subgroup U under $< a > \cap U =< 1 >$ and $L\ \{a\} \subseteq U$. Let T/U be the periodic part of A/U. By the selection of U an element $< aU >$ has infinite order, so that $a \notin T$. It follows that $U = T$, so that A/U is torsion-free. In other words, U is a pure subgroup of A. Since U does not include C, U is finitely generated. By the selection of U every non-identity cyclic subgroup of A/U has a non-identity intersection with $< aU >$. In other words, A/U is a locally cyclic group (that is $r_0(A/U) = 1$). Since a finitely generated subgroup of A has finite 0-rank, A has a finite 0-rank. □

Theorem 3. Let G be a non-abelian locally nilpotent torsion-free group of finite 0-rank. Suppose that G is not finitely generated but $[G,G]$ is finitely generated. Every non polycyclic subgroup of G is nearly normal if and only if G satisfies the following conditions:

(1) G includes a finitely generated subgroup E such that $G = E\zeta(G)$ and $E \cap \zeta(G) = [G,G]$;

(2) $\zeta(G)$ is not finitely generated;

(3) $[G,G]$ $\mathbf{PM}_{fg}(\zeta(G))$, in particular, $\mathbf{PM}_{fg}(\zeta(G)) \neq < 1 >$, therefore $\zeta(G)$ includes a finitely generated pure subgroup M such that $\zeta(G)/M$ is a non-cyclic locally cyclic group.

Proof. By Lemma 8 $Z = \zeta(G)$ is not finitely generated. Corollary to Lemma 8 yields that G/Z is abelian and finitely generated. In particular, $K = [G,G] \leq \zeta(G)$. An abelian group G/K has a free abelian factor-group $(G/K)/(Z/K)$, therefore $G/K = Z/K \times E/K$ where $G/Z \cong E/K$ is a finitely generated abelian group (see, for example, [15, Theorem 14.4]). Together with K a subgroup E is finitely generated. Finally, if S is a pure subgroup of Z such that S is not finitely generated, then Lemma 1 yields that G/S has finite derived subgroup. On the other hand, Z/S is torsion-free and also G/Z is torsion-free (see, for example, [16, Theorem 2.25]), so that G/S is torsion-free. Hence this factor-group is abelian, that is $K \leq S$. It implies that $[G,G] \leq \mathbf{PM}_{fg}(\zeta(G))$. Finally, an application of Lemma 9 concludes a proof of (3).

Conversely, suppose that G satisfies the conditions (1) — (3). Let H be a not polycyclic subgroup of G. Since G/Z is finitely generated, $H \cap Z$ can not be finitely generated. Let L be a pure envelope of a subgroup $H \cap Z$ in Z, then L is not finitely generated. By (3) $[G,G] = K \leq L$. Since $[G,G] \leq Z$, $K \cap H = K \cap H \cap Z$. Using

$$K/(K \cap H) = K/(K \cap H \cap Z) \cong K(H \cap Z)/(H \cap Z) \leq L/(H \cap Z),$$

And the fact that $L/(H \cap Z)$ is periodic, we obtain that $K/(K \cap H)$ is also periodic. On the other hand, K is finitely generated, so its periodic factor-group $K/(K \cap H)$ is finite. Thus $G/(K \cap H)$ has a finite derived subgroup $K/(K \cap H)$. Then every its subgroup is nearly normal, in particular, so is $H/(K \cap H)$. It follows that H is nearly normal in G. □

Lemma 10. Let G be a non-abelian locally nilpotent torsion-free group of finite 0-rank. Suppose that $[G,G]$ is not finitely generated. If every non polycyclic subgroup of G is nearly normal, then $\zeta(G)$ includes a finitely generated subgroup M such that $\zeta(G)/M$ is a Prüfer p-subgroup for some prime p.

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Proof. By Lemma 8 $Z = \zeta(G)$ is not finitely generated and $K = [G, G] \leq \zeta(G)$. Since G has a finite 0-rank, there exists a finitely generated subgroup E of Z such that Z/E is periodic. If a set $\Pi(Z/E)$ is finite, then there are two infinite subsets Δ and Ξ such that $\Delta \cap \Xi = \Pi(Z/E)$ and $\Delta \cap \Xi = \oslash$. Let D/E be a Sylow Ξ-subgroup of Z/E and S/E be a Sylow Ξ-subgroup of Z/E, then $Z = DS$ and $D \cap S = E$. Every subgroup D and S are not polycyclic, therefore by Lemma 1 $[G/D, G/D]$ and $[G/S, G/S]$ are finite. From $E = D \cap S$ we obtain embedding $G/E \hookrightarrow G/D \times G/S$, which proves that $[G/E, G/E]$ is finite. Since E is finitely generated, $[G, G]$ is likewise finitely generated. This contradiction shows that a set $\Pi(Z/E)$ is finite. In turn, it implies that Z/E is a Chernikov group. Let P/E be a divisible part of Z/E. If it is not a Pr$\ddot{u}$fer subgroup, then $P/E = U/E \cap V/E$ where the both subgroups U/E are V/E non-identity and divisible, in particular, they are not polycyclic. By Lemma 1 $[G/U, G/U]$ and $[G/V, G/V]$ are finite. From $E = U \cap V$ we obtain embedding $G/E \hookrightarrow G/U \times G/V$, which proves that $[G/E, G/E]$ is finite. Thus we come again to a contradiction. This contradiction shows that P/E is a Pr$\ddot{u}$fer subgroup. Since P/E is divisible, $Z/E = P/E \cap M/E$ where a subgroup M/E is finite. Then M is finitely generated and Z/M is a Pr$\ddot{u}$fer p-subgroup. □

Corollary. Let G be a non-abelian locally nilpotent torsion-free group of finite 0-rank. Suppose that $[G, G]$ is not finitely generated. If every non polycyclic subgroup of G is nearly normal, then $G/\zeta(G)$ is an abelian minimax group, moreover, $\mathbf{Sp}(G/\zeta(G)) = \mathbf{Sp}(\zeta(G)) = \{p\}$ where p is a prime.

Proof. By Lemma 8 $\zeta(G)$ is not finitely generated and $G/\zeta(G)$ is abelian. Hence for each element $g \in G$ the mapping $\phi_g : x \longrightarrow [x, g]$, $x \in G$, is an endomorphism, so that

$$[G, g] = \mathbf{Im}\phi_g \cong A/\mathbf{Ker}\phi_g = G/C_G(g).$$

An inclusion $[G, G] \leq \zeta(G)$ and lemma 10 show that $[G, g]$ is minimax and $\mathbf{Sp}([G, g]) = \{p\}$ where p is a prime. Then $G/C_G(g)$ is likewise minimax and $\mathbf{Sp}(G/C_G(g)) = \{p\}$ for each element $g \in G$. Let $\{g_1\zeta(G), \ldots, g_t\zeta(G)\}$ be a maximal $\mathbf{Z}$-independent set of $G/\zeta(G)$, $H =< g_1, \ldots, g_t >$. Using again a Remak's theorem we obtain an embedding $G/C_G(H) \hookrightarrow G/C_G(g_1) \times \ldots \times G/C_G(g_t)$, which proves that $G/C_G(H)$ is minimax and $\mathbf{Sp}(G/C_G(H)) = \{p\}$.

Put $D = H\zeta(G)$, then $C_G(H) = C_G(D)$, in particular, G$/C_G(D)$ is minimax and $\mathbf{Sp}(G/C_G(D)) = \{p\}$. By the selection of D for every element $g \in G$ there is a positive integer m such that $g^m \in D$. We have

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for each element $x \in C_G(D)$ an equation $g^m x = x g^m$. The last equation implies in locally nilpotent torsion-free group G an equation $gx = xg$. In turn it implies that $C^G(D) \leq \zeta(G)$, that is $C_G(D) = \zeta(G)$. Hence $G/\zeta(G)$ is minimax and $\mathbf{Sp}(G/\zeta(G)) = \{p\}$. □

Lemma 11. Let G be a non-abelian locally nilpotent torsion-free group of finite 0-rank. Suppose that $[G, G]$ is not finitely generated. If every non polycyclic subgroup of G is nearly normal, then a section $A/(A \cap [G, G])$ is finitely generated for each abelian subgroup A.

Proof. Suppose the contrary, let $A/(A \cap [G, G])$ is not finitely generated. By Lemma 8 $K = [G, G] \leq \zeta(G)$, so that AK is abelian. Therefore we may assume that $K \leq A$. Let E be a finitely generated subgroup of K such that K/E is a Prüfer p-subgroup. Such subgroup exists by Lemma 10. By the properties of the divisible subgroups in abelian groups (see., for example, [15, Theorem 21.2]) $A/E = K/E \times L/E$ for some subgroup L. By our assumption about A/K a subgroup L can not be finitely generated. Then it is nearly normal. Lemma 4 yields that L includes a G-invariant torsion-free subgroup B of finite index. In particular, B is not finitely generated. By Lemma 1 $[G/K, G/K]$ and $[G/B, G/B]$ are finite. We have again $G/(K \cap B) \hookrightarrow G/K \times G/B$, what implies that $G/(K \cap B)$ has a finite derived subgroup. Together with $K \cap B \leq K \cap L = E$ it follows that $[G, G]$ is finitely generated. This contradiction shows that A/K is finitely generated. □

Lemma 12. Let G be a non-abelian locally nilpotent torsion-free group of finite 0-rank. Suppose that $[G, G]$ is not finitely generated. If every non polycyclic subgroup of G is nearly normal, then an intersection $A \cap [G, G]$ is not finitely generated whenever A is not finitely generated.

Proof. Suppose the contrary, let $A \cap [G, G]$ is finitely generated. By Lemma 4 A includes a G-invariant torsion-free subgroup C of finite index (remark, that A is not polycyclic, so that it is nearly normal). In particular, C can not be polycyclic, and Lemma 1 shows that $[G/C, G/C]$ is finite. Usin an embedding $G/(C \cap [G, G]) \hookrightarrow G/[G, G] \times G/C$, we can prove that $G/(C \cap [G, G])$ has a finite derived subgroup. Since $C \cap [G, G] \leq A \cap [G, G]$, $C \cap [G, G]$ is finitely generated. It follows that $[G, G]$ is likewise finitely generated. This contradiction shows that $A \cap [G, G]$ is not finitely generated. □

Corollary. Let G be a non-abelian locally nilpotent torsion-free group of finite 0-rank. Suppose that $[G, G]$ is not finitely generated. If every non polycyclic subgroup of G is nearly normal, then an intersection $A \cap [G, G]$ has finite index in $[G, G]$ whenever A is not finitely generated.

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In fact, by Lemma 12 an intersection $A \cap [G, G]$ can not be polycyclic. By Lemma 1 $G/(A \cap [G, G])$ has a finite derived subgroup. This means that $A \cap [G, G]$ has a finite index in $[G, G]$.

Theorem 4. Let G be a non-abelian locally nilpotent torsion-free group of finite 0-rank. Suppose that $[G, G]$ is not finitely generated. Every non polycyclic subgroup of G is nearly normal if and only if G satisfies the following conditions:

(1) $\zeta(G)$ includes a finitely generated subgroup M such that $\zeta(G)/M$ is a Prüfer p-subgroup for some prime p;

(2) $G/\zeta(G)$ is an abelian minimax group such that $\mathbf{Sp}(G/\zeta(G)) = \mathbf{Sp}(\zeta(G)) = \{p\}$;

(3) $[G, G] \le \mathbf{PM}_{fg}(\zeta(G))$, in particular, $\mathbf{PM}_{fg}(\zeta(G)) \neq < 1 >$;

(4) if A is an abelian subgroup of G then a section $A/(A \cap [G, G])$ is finitely generated. Furthermore, if a subgroup H is not finitely generated, then an intersection $H \cap [G, G]$ has finite index in $[G, G]$.

Proof. A condition (1) follows from Lemma 10. A condition (2) follows from Corollary to Lemma 10. Corollary to Lemma 8 proves an inclusion $[G, G] \le \zeta(G)$, which implies (3). Finally, a condition (4) follows from Lemma 11 and a last statement follows from Corollary to Lemma 12.

Conversely, suppose that a group G satisfies all conditions (1) —- (4). Let H be a non polycyclic subgroup of G. Then H includes an abelian subgroup A such that A is not finitely generated. By (4) an intersection $A \cap [G, G]$ is not finitely generated. Let L be a pure envelope of $B = A \cap [G, G]$ in $Z = \zeta(G)$. Then L is not polycyclic, and a condition (3) implies an inclusion $[G, G] \le L$. Since L/B is periodic, the conditions (3) and (1) show that L/B is finite. By $[G, G]B/B \le L/B$ we obtain now that $[G/B, G/B]$ is finite. Then every subgroup of G/B is nearly normal, in particular, so is and H/B. It follows that H is nearly normal in G. □

Consequently, now we must consider a case, when G is not locally nilpotent and all periodic subgroups of G are finite. In particular, $\mathbf{P}(G)$ is finite. Here $\mathbf{P}(G)$ is a maximal normal periodic subgroup of G. Proposition 1 make possible the next reduction $\mathbf{P}(G) =< 1 >$. Since $G \in \mathfrak{S}_1\mathfrak{F}$, G includes the normal subgroups H, A such that $H \le A$, H is nilpotent and torsion-free, A/H is finitely generated free abelian group and G/A is finite [20, Theorem 6]).

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Lemma 13. Let $G \in \mathfrak{S}_1\mathfrak{F}$ and suppose that $\mathbf{P}(G) =< 1 >$ and $[G, G]$ is infinite. If every non polycyclic-by-finite subgroup of G is nearly normal, then G includes an abelian torsion-free normal subgroup A such that A is not finitely generated and G/A has a finite derived subgroup and finite periodic part.

Proof. As we have noted above, G includes the normal subgroups H, L such that $H \leq L$, H is nilpotent and torsion-free, L/H is finitely generated free abelian group and G/L is finite. Since G/H is polycyclic-by-finite, H can not be polycyclic. By Lemma 8 $A = \zeta(H)$ is not finitely generated. Lemma 1 yields that $[G/A, G/A]$ is finite. Since H/A and L/H are torsion-free and G/L is finite, the periodic part of G/A is finite. □

Lemma 14. Let $G \in \mathfrak{S}_1\mathfrak{F}$ and suppose that $\mathbf{P}(G) =< 1 >$ and $[G, G]$ is infinite but polycyclic-by-finite. If every non polycyclic-by-finite subgroup of G is nearly normal, then G is a nilpotent torsion-free group.

Proof. As we have noted above, G includes the normal subgroups H, L such that $H \cap L$, H is nilpotent and torsion-free, L/H is finitely generated free abelian group and G/L is finite. Since $[G, G]$ is polycyclic-by-finite, $[H, H]$ is likewise polycyclic. By Corollary to Lemma 8 $A = \zeta(H)$ is not finitely generated but H/A is abelian and finitely generated. Since G/H is polycyclic-by-finite, G/A is polycyclic-by-finite. Lemma 1 yields that G/A has a finite derived subgroup. For each element $g \in G$ the mapping $\phi_g : a \longrightarrow [a, g]$, $a \in A$, is an endomorphism, so that $\mathbf{Ker}\phi_g = C_A(g)$ and $\mathbf{Im}\phi_g = [A, g]$ are the $< g >$-invariant subgroups and $[A, g] = \mathbf{Im}\phi_g \cong A/\mathbf{Ker}\phi_g = A/C_A(g)$. Together with $[G, G]$ a subgroup $[A, g]$ is finitely generated, thus $E = C_A(g)$ is not polycyclic, in particular, it is nearly normal.. Let $G/A =< g_1A, \ldots, g_nA >$. Since $E_1 = C_A(g_1)$ is not polycyclic, it is nearly normal. By Lemma 4 E_1 includes a G-invariant subgroup B_1 of finite index. In particular, B_1 is not polycyclic. Repeating the above arguments, we obtain that $B_2 = B_1 \cap C_A(g_2)$ is not polycyclic. Using the same arguments after finitely many steps we obtain that $C = C_A(g_1, \ldots, g_n)$ is not polycyclic. By Lemma 1 $[G/C, G/C]$ is finite. Hence the set T/C of all elements having finite orders is a subgroup. A subgroup $[T, T]$ is locally finite (see, for example, [16, Corollary to Theorem 4.12]). In particular, the set P of elements of T having finite orders, is a (characteristic) subgroup of T. Moreover, T/P is abelian torsion-free group. An equation $\mathbf{P}(G) =< 1 >$ shows that T is an abelian torsion-free subgroup. This together with inclusion $C_A(g_1, \ldots, g_n) \leq \zeta(G)$ imply that $T \leq \zeta(G)$, because T/C is periodic. Since G/T is abelian, G is nilpotent. Finally, an equation $\mathbf{P}(G) =< 1 >$ shows that G is torsion-free. □

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Corollary. Let G be an abelian-by-finite group, whose non polycyclic-by-finite subgroups are nearly normal. Suppose that $\mathbf{P}(G) =< 1 >$. If G is non- abelian, then $[G, G]$ is not finitely generated.

Proof. Suppose the contrary, let $[G, G]$ is finitely generated. By Lemma 14 G is nilpotent. In particular, it is torsion-free. Since G is abelian-by-finite, G is abelian. This contradiction proves that $[G, G]$ is not polycyclic-by-finite. □

Lemma 15. Let $G \in \mathfrak{S}_1\mathfrak{F}$ and suppose that $\mathbf{P}(G) =< 1 >$ and $[G, G]$ is not polycyclic-by-finite. If every non polycyclic-by-finite subgroup of G is nearly normal, then every abelian torsion-free subgroup A includes a finitely generated subgroup B such that A/B is a Prüfer p-subgroup for some prime p.

Proof. Since $G \in \mathfrak{S}_1\mathfrak{F}$, A has a finite 0-rank. Then A includes a finitely generated subgroup E such that A/E is periodic. If a set $\Pi(A/E)$ is infinite, then there are two infinite subsets Δ and Ξ such that $\Delta \cup \Xi = \Pi(Z/E)$ and $\Delta \cap \Xi = \oslash$. Let D/E be a Sylow Ξ-subgroup and S/E be a Sylow Ξ-subgroup of A/E. Then $Z = DS$ and $D \cap S = E$. The both subgroups D and S are not polycyclic-by-finite, so that $[G/D, G/D]$ and $[G/S, G/S]$ are finite by Lemma 1. An equation $E = D \cap S$ gives an embedding $G/E \hookrightarrow G/D \times G/S$, which shows that $[G/E, G/E]$ is finite. Since E is finitely generated, $[G, G]$ is finitely generated. Thus we obtain a contradiction with Corollary of Lemma 14. This contradiction proves that $\Pi(A/E)$ is finite. This means that A/E is a Chernikov group. Let P/E be a divisible part of A/E. If we suppose, that it is not a Prüfer subgroup, then $P/E = U/E \times V/E$ where U/E and V/E are non-identity and divisible. In particular, U and V are not polycyclic. By Lemma 1 $[G/U, G/U]$ and $[G/V, G/V]$ are finite. From $E = U \cap V$ we obtain an embedding $G/E \hookrightarrow G/U \times G/V$, which shows that G/E has finite derived subgroup. This come again to contradiction, which prove that P/E is a Prüfer subgroup. Since P/E is divisible, $Z/E = P/E \times B/E$ where B/E is finite. Then B is finitely generated and A/B is a Prüfer subgroup. □

Corollary 1. Let $G \in \mathfrak{S}_1\mathfrak{F}$ and suppose that $\mathbf{P}(G) =< 1 >$ and $[G, G]$ is not polycyclic-by-finite. If every non polycyclic-by-finite subgroup of G is nearly normal, then $[G, G]$ satisfies the following conditions:

(1) $[G, G]$ includes a G-invariant abelian torsion-free subgroup A of finite index;

(2) A includes a finitely generated subgroup B such that A/B is a Prüfer p-subgroup for some prime p;

(3) if C is a subgroup of A and $r_0(C) < r_0(A)$ then C is finitely generated.

Proof. Since $[G, G]$ is not polycyclic-by-finite, so is G. Then G includes an abelian subgroup U, which is not finitely generated (see, for example, [16, Theorem 3.31]). By $\mathbf{P}(G) =< 1 >$ a periodic part of U is finite. Without loss of generality we can suppose that U is torsion-free. Among the not polycyclic-by-finite subgroups of U choose a subgroup B having the biggest 0-rank. It is possible, because the 0-rank of U is finite. Since B is not polycyclic-by-finite, B is nearly normal. By Lemma 4 B includes a G-invariant subgroup E of finite index. In particular, E is not polycyclic-by-finite. By Lemma 1 $[G/E, G/E]$ is finite. Put $A = [G, G] \cap E$. Then from $[G, G]/A \cong [G, G]/([G, G] \cap E) \cong [G, G]E/E$ we obtain that $[G, G]/A$ is finite. Since $[G, G]$ is not polycyclic-by-finite, A is not finitely generated. Therefore $r_0(U) = r_0(A)$, and from a selection of U follows (3). Finally, (2) follows from Lemma 15. $\square$

Corollary 2. Let $G \in \mathfrak{S}_1\mathfrak{F}$ and suppose that $\mathbf{P}(G) =< 1 >$ and $[G, G]$ is not polycyclic-by-finite. Assume also that every non polycyclic-by-finite subgroup of G is nearly normal. If A is a G-invariant abelian torsion-free subgroup of $[G, G]$ having finite index, then either $C_A(g) = A$ or $C_A(g) =< 1 >$ for each element $g \in G$.

Proof. Use again an endomorphism $\phi_g : a \longrightarrow [a, g]$, $a \in A$. We have $\mathbf{Ker}\phi_g = C_A(g)$ and $\mathbf{Im}\phi_g = [A, g]$, so that $[A, g] = \mathbf{Im}\phi_g \cong A/\mathbf{Ker}\phi_g = A/C_A(g)$. It is not hard to see, that $C_A(g)$ is a pure subgroup of A. If we suppose that $A \neq C_A(g) \neq< 1 >$, then $r_0(C_A(g)) < r_0(A)$ and a condition (3) of Corollary 1 to Lemma 15 shows that $C_A(g)$ is finitely generated. Similarly we obtain that $r_0([A, g]) < r_0(A)$, and using the same arguments we obtain that $[A, g]$ is finitely generated. An isomorphism $[A, g] \cong A/C_A(g)$ shows that A is finitely generated. Hence $[G, G]$ is polycyclic-by-finite. This contradiction shows that $A \neq C_A(g)$ is possible only whenever $C_A(g) =< 1 >$. $\square$

Lemma 16. Let $G \in \mathfrak{S}_1\mathfrak{F}$ and suppose that $\mathbf{P}(G) =< 1 >$ and $[G, G]$ is not polycyclic-by-finite. Assume also that every non polycyclic-by-finite subgroup of G is nearly normal. If A is a G-invariant abelian torsion-free subgroup of $[G, G]$ such that index $/[G, G] : A/$ is finite and $A \leq \zeta(G)$, then G is nilpotent.

Proof. By Lemma 1 $[G/A, G/A]$ is finite. Therefore the set T/A of all elements having finite orders is a subgroup. Then $[T, T]$ is a locally finite subgroup (see, for example, [16, Corollary to Theorem 4.12]). In particular, a set P of all elements of T having finite orders is a (characteristic) subgroup of T. Moreover, T/P is an abelian torsion-free group. By $\mathbf{P}(G) =< 1 >$ we obtain that T is an abelian torsion-free subgroup. Since T/A is periodic, an inclusion $A \leq \zeta(G)$ yields $T \leq \zeta(G)$. Then G is nilpotent because G/T is abelian. □

Lemma 17. Let $G \in \mathfrak{S}_1\mathfrak{F}$ and suppose that $\mathbf{P}(G) =< 1 >$ and $[G, G]$ is not polycyclic-by-finite. Assume also that every non polycyclic-by-finite subgroup of G is nearly normal. If A is a G-invariant abelian torsion-free subgroup of $[G, G]$ such that index $/[G, G] : A/$ is finite and there is an element $g \in G$ with a property $C_A(g) \cap A$, then G includes a subgroup $D = AV$ of finite index such that $A \cap V =< 1 >$ and V is an abelian torsion-free subgroup.

Proof. By Corollary 2 of Lemma 15 $C_A(g) =< 1 >$. Lemma 1 yields that $[G/A, G/A]$ is finite. Then $L/A = C_{G/A}(gA)$ has finite index in G/A. This together with $L/A \in \mathfrak{S}_1\mathfrak{F}$, implies that it includes an abelian subgroup M/A of finite index. Since $gA \in \zeta(L/A)$, $< gA, M/A >$ is abelian. Thus we may assume that L/A is abelian. Using an isomorphism $[A, g] \cong A/C_A(g) \cong A$ and a condition (2) of Corollary 1 to Lemma 15 we obtain that $A/[A, g]$ is finite. It follows that $C = C_L(A/[A, g])$ has in L a finite index. Let $k = /A/[A, g]/$ and $E = Ck$. Since $C \in \mathfrak{S}_1\mathfrak{F}$, C/E is finite, and therefore L/E is likewise finite. Let $c \in C$. We have $[c^2, g] = [c, g]^c[c, g] = [c, g^c][c, g]$. Since $g^c = ga$ for some element $a \in A$, $[c, ga] = [c, g]^a[c, a]$, thus $[c^2, g] = [c, g]^2[c, a]$ where $[c, a] = v \in [A, g]$. With the help of similarly arguments we obtain that $[c^k, g] = [c, g]^k w$ for some element $w \in [A, g]$. From $[c, g \in A$ we obtain $[c, g]^k \in A^k \leq [A, g]$, therefore $[c^k, g] \in [A, g]$, that is $[c^k, g] = [b, g]$ for some element $b \in A$. It follows that $c^k b^{-1} \in C_L(g)$ or $c^k \in C_L(g)A$. In other words, we obtain an inclusion $E \leq C_L(g)A$. An equation $C_A(g) =< 1 >$ implies $C_L(g) \cap A =< 1 >$, thus we have $C_L(g) \cong C_L(g)/(C_L(g) \cap A) \cong C_L(g)A/A \geq EA/A$. As we remarked, EA/A is abelian and has finite index in G/A. Hence $V = C_L(g)$ is abelian and $D = AE$ has finite index in G. □

Corollary. Let $G \in \mathfrak{S}_1\mathfrak{F}$ and suppose that $\mathbf{P}(G) =< 1 >$ and the both group $[G, G]$ and $G/[G, G]$ are not polycyclic-by-finite. Assume also that every non polycyclic-by-finite subgroup of G is nearly normal. Then G is nilpotent.

Proof. By Corollary 1 of Lemma 15 $[G, G]$ includes a G-invariant abelian torsion-free subgroup A having finite index. If $C_A(g) = A$ for each element $g \in G$, then $A \leq \zeta(G)$. By Lemma 16 G is nilpotent. Consequently we may assume that there is an element g with a property $C_A(g) \neq A$.

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By Lemma 17 G includes a subgroup $D = AV$ of finite index such that $A \cap V =< 1 >$ and V is abelian torsion-free subgroup. In particular, V is not finitely generated and hence is nearly normal. Lemma 4 proves that there exists a G-invariant abelian subgroup B having finite index in V. By Lemma 1 $[G/B, G/B]$ is finite. From $A \cap B \leq A \cap V =< 1 >$ we obtain an embedding $G \hookrightarrow G/A \times G/B$, which proves that $[G, G]$ is finite. This contradiction shows that $C_A(g) = A$ for each element $g \in G$. As we see above it implies that G is nilpotent. □

Now we can obtain a description and for the last case.

Theorem 5. Let $G \in \mathfrak{S}_1\mathfrak{F}$ and suppose that G is not polycyclic-by-finite, $\mathbf{P}(G) =< 1 >$ and G is not nilpotent. Every non polycyclic-by-finite subgroup of G is nearly normal if and only if G satisfies the following conditions:

(1) $[G, G]$ includes a G-invariant abelian torsion-free subgroup of finite index;

(2) A includes a finitely generated subgroup B such that A/B is a Prüfer p-subgroup for some prime p;

(3) if C is a subgroup of A and $r_0(C) < r_0(A)$, then C is finitely generated;

(4) $C_G(A) \neq G$ and $C_A(g) =< 1 >$ for each element $g \in G$ $C_G(A)$;

(5) G includes a subgroup $D = AV$ of finite index such that $A \cap V =< 1 >$ and V is a finitely generated torsion-free abelian subgroup.

Proof. By Lemma 14 $[G, G]$ is not polycyclic-by-finite. Corollary 1 of Lemma 15 shows that G satisfies (1) —- (3). A condition (4) follows from Lemma 16. By Corollary to Lemma 17 $G/[G, G]$ is finitely generated. Finally, using Lemma 17 we obtain that G satisfies (5).

Conversely, suppose that G satisfies the conditions (1) —- (5). Let H be an arbitrary non polycyclic-by-finite subgroup of G. Since $G/[G, G]$ is finitely generated, an intersection $H \cap [G, G]$ can not be polycyclic-by-finite. By (3) we obtain that $r_0(H \cap A) = r_0(A)$. It follows that $H \cap A$ has a finite index s in a subgroup A. Then $B = A^s \leq H \cap A$. Clearly B is a normal subgroup of G. Since $r_0(A)$ is finite, A/B is finite. This implies that G/B has a finite derived subgroup. Then every subgroup of G/B is nearly normal, in particular, so is H/B. Hence and H is nearly normal in G. □

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