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RESEARCH ARTICLE

On autotopies and automorphisms of n-ary linear quasigroups

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Abstract. In this article we study structure of autotopies, automorphisms, autotopy groups and automorphism groups of n-ary linear quasigroups.

We find a connection between automorphism groups of some special kinds of n-ary quasigroups (idempotent quasigroups, loops) and some isotopes of these quasigroups. In binary case we find more detailed connections between automorphism group of a loop and automorphism group of some its isotope. We prove that every finite medial n-ary quasigroup of order greater than 2 has a non-identity automorphism group.

We apply obtained results to give some information on automorphism groups of n-ary quasigroups that correspond to the ISSN code, the EAN code and the UPC code.

1. Introduction

We shall use basic terms and concepts from books [1], [2], [3], [15], [6]. To give some n-ary definitions we take into consideration articles [23], [22], [12].

We recall some known facts. Let Q be a nonempty set, let n be natural number, $n \geq 2$. A map f that maps all n-tuples over Q into elements of the set Q is called *n-ary operation*, i.e. $f(x_1, x_2, \dots, x_n) = x_{n+1}$ for all $(x_1, x_2, \dots, x_n) \in Q^n$ and $x_{n+1} \in Q$.

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A nonempty set Q with n-ary operation f such that in the equation $f(x_1, x_2, \dots, x_n) = x_{n+1}$ knowledge of any n elements of $x_1, x_2, \dots, x_n$, x_{n+1} uniquely specifies the remaining one is called an *n-ary quasigroup* ([3]).

We say that n-ary quasigroup (Q, f) is an *isotope of n-ary quasigroup* (Q, g) if there exist permutations $\mu_1, \mu_2, \dots, \mu_n, \mu$ of the set Q such that

$$f(x_1, x_2, \dots, x_n) = \mu^{-1} g(\mu_1 x_1, \dots, \mu_n x_n) \qquad (1)$$

for all $x_1, \dots, x_n \in Q$. We can write this fact also in the form $(Q, f) = (Q, g)T$ where $T = (\mu_1, \mu_2, \dots, \mu_n, \mu)$.

If in (1) $f = g$, then $(n+1)$-tuple $(\mu_1, \mu_2, \dots, \mu_n, \mu)$ of permutations of the set Q is called an *autotopy of n-quasigroup* (Q, f). The last component of an autotopy of an n-quasigroup is called a *quasiautomorphism* (by analogy with binary case).

A set of all autotopies of a quasigroup (Q, f) forms the group of autotopies relatively the usually defined operation on this set: if $T_1 = (\mu_1, \mu_2, \dots, \mu_n, \mu)$ and $T_2 = (\nu_1, \nu_2, \dots, \nu_n, \nu)$ are autotopies of quasigroup (Q, f), then $T_1 \circ T_2 = (\mu_1\nu_1, \mu_2\nu_2, \dots, \mu_n\nu_n, \mu\nu)$ is an autotopy of quasigroup (Q, f).The autotopy group of a quasigroup (Q, f) will be denoted as $\mathfrak{T}(Q, f)$.

If in (1) $\mu_1 = \mu_2 = \dots = \mu_n = \mu$, then quasigroups (Q, f) and (Q, g) are isomorphic.

At last, if in (1) the n-ary operations f and g are equal and $\mu_1 = \mu_2 = \dots = \mu_n = \mu$, then we obtain an *automorphism of quasigroup* (Q, f), i.e. a permutation μ of the set Q is called an automorphism of an n-quasigroup (Q, f) if for all $x_1, \dots, x_n \in Q$ the following relation is fulfilled: $\mu f(x_1, \dots, x_n) = f(\mu x_1, \dots, \mu x_n)$. We denote by $Aut(Q, f)$ the automorphism group of an n-ary quasigroup (Q, f).

A sequence $x_m, x_{m+1}, \dots, x_n$, where m, n are natural numbers and $m \leq n$, will be denoted by x_m^n, a sequence $x, \dots, x$ (k times) will be denoted by $\overline{x}^k$. The expression $\overline{1, n}$ designates a set $\{1, 2, \dots, n\}$ of natural numbers ([3]).

As usual, $L_a^\circ : L_a^\circ x = a \circ x$, $R_a^\circ : R_a^\circ x = x \circ a$ are respectively left and right translations of binary quasigroup $(Q, \circ)$. We shall omit denotation of a quasigroup operation by using of quasigroup translations, i.e. we shall write L_a, R_a instead of L_a°, R_a° in cases when it will be clear from context relatively which quasigroup operation we take quasigroup translations.

$M(Q, \cdot)$ denotes a group generated by all left and right translations of a binary quasigroup $(Q, \cdot)$ and it is called the *multiplication group of a quasigroup* $(Q, \cdot)$.

We shall denote the identity permutation as ε, the order of a set Q as $|Q|$.

We shall use the following theorem ([3]).

Theorem 1. *If n-ary quasigroups (Q, f) and (Q, g) are isotopic with isotopy T, i.e. $(Q, f) = (Q, g)T$, then $\mathfrak{I}(Q, f) = T^{-1}\mathfrak{I}(Q, g)T$.*

Remark 1. In algebra usually quasigroups are studied up to isomorphism. Thus, without loss of a generality, any isotopy T of an n-ary quasigroup (Q, g) we can choose in such manner that its last component is the identity map.

An n-ary quasigroup (Q, g) of the form $\gamma g(x_1, x_2, \dots, x_n) = \gamma_1 x_1 + \gamma_2 x_2 + \dots + \gamma_n x_n$, where $(Q, +)$ is a group, $\gamma, \gamma_1, \dots, \gamma_n$ are some permutations of the set Q, is called an *n-ary group isotope*. This equality (as well as analogous equalities that will appear later in this article) is true for all $x_1, x_2, \dots, x_n \in Q$.

An n-quasigroup (Q, g) of the form $g(x_1, x_2, \dots, x_n) = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n + a = \sum_{i=1}^{n} \alpha_i x_i + a$, where $(Q, +)$ is a group, $\alpha_1, \dots, \alpha_n$ are some automorphisms of the group $(Q, +)$, the element a is some fixed element of the set Q, will be called a *linear n-ary quasigroup* (over group $(Q, +)$).

An n-ary linear quasigroup (Q, g) over an abelian group $(Q, +)$ is called *n-T-quasigroup ([23])*. If $n = 2$, then a quasigroup from this quasigroup class is called a *T-quasigroup* ([14, 11]).

The following identity of n-ary quasigroup (Q, g)

$$\begin{aligned} &g(g(x_{11}, x_{12}, \dots, x_{1n}), g(x_{21}, x_{22}, \dots, x_{2n}), \dots, g(x_{n1}, x_{n2}, \dots, x_{nn})) = \\ &g(g(x_{11}, x_{21}, \dots, x_{n1}), g(x_{12}, x_{22}, \dots, x_{n2}), \dots, g(x_{1n}, x_{2n}, \dots, x_{nn})) \end{aligned} \tag{2}$$

is called medial identity ([3]). An n-ary quasigroup with identity (2) is called *medial n-ary quasigroup*.

In binary case from identity (2) we obtain usual *medial identity*: $xy \cdot uv = xu \cdot yv$.

In [3] V.D. Belousov proved the following theorem.

Theorem 2. *Let (Q, f) be a medial n-quasigroup. Then there exist an abelian group $(Q, +)$, its commuting in pairs automorphisms $\alpha_1, \dots, \alpha_n$, and a fixed element a of the set Q such that $f(x_1, x_2, \dots, x_n) = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n + a = \sum_{i=1}^{n} \alpha_i x_i + a$ for all $x_i \in Q$, $i \in \overline{1, n}$.*

In binary case from Belousov theorem it follows classical Toyoda theorem (T-theorem) ([24], [13], [4], [1], [2]).

Medial quasigroups, as well as the other classes of quasigroups isotopic to groups, give us a possibility to construct quasigroups with preassigned properties. Often it is possible to express these properties on the language of properties of groups and components of isotopy. Systematically this approach was used by study of T-quasigroups in [8], [9], [14], [11].

Automorphisms and automorphism groups of some binary and n-ary quasigroups were studied in many articles, see, for example, [13], [9], [8], [16], [18], [5], [19], [22], [7], [12].

Our approach to study the automorphisms of n-ary linear quasigroups is based on the very known fact that isotopic quasigroups have isomorphic groups of autotopies (Theorem 1).

The main idea is that any automorphism is an autotopy with equal components. So, if we know the structure of autotopies of an n-quasigroup (Q, f) and the form of isotopy T, then we have a possibility to obtain information on autotopies and automorphisms of n-quasigroup $(Q, g) = (Q, f)T$. In binary case this observation was used by study of the automorphisms of some quasigroup classes ([17]), [18], [19], [22], [7]).

2. On autotopies and automorphisms of n-ary linear quasigroups

In this section we study structure of autotopies, automorphisms, autotopy groups and automorphism groups of some classes of n-ary group isotopes.

We shall use the following elementary properties of quasigroups.

Lemma 1. *(i) If $(Q, \cdot)$ is a binary quasigroup, L_a, R_b are some its left and right translations, $\varphi \in Aut(Q, \cdot)$, then $\varphi L_a = L_{\varphi a}\varphi$, $\varphi R_b = R_{\varphi b}\varphi$.*

(ii) If $(Q, +)$ is a group, then $L_a R_b = R_b L_a$, $L_a^{-1} = L_{-a}, R_a^{-1} = R_{-a}$.

(iii) If $(Q, +)$ is a group, then $R_d = L_d I_d$, where I_d is the inner automorphism of the group $(Q, +)$, i.e. $I_d x = -d + x + d$ for all $x \in Q$.

(iv) Any quasiautomorphism of a group $(Q, +)$ has the form $L_a\beta$, where $a \in Q$, $\beta \in Aut(Q, +)$.

Proof. (i) We have $\varphi L_a x = \varphi(a\cdot x) = \varphi a\cdot\varphi x = L_{\varphi a}\varphi x$, $\varphi R_b x = \varphi(x\cdot b) = \varphi x \cdot \varphi b = R_{\varphi b}\varphi x$.

(ii) $L_a R_b x = a + (x + b) = (a + x) + b = R_b L_a x$. $L_a^{-1} = L_{-a}$ since $L_a^{-1} L_a x = x = -a + a + x = L_{-a} L_a x$.

(iii) $R_d x = x + d = d - d + x + d = L_d I_d x$.

(iv) Any autotopy of a group $(Q, +)$ has the form $(L_c\theta, R_d\theta, L_c R_d\theta)$ where $\theta \in Aut(Q, +)$, L_c is a left and R_d is a right translation of the group $(Q, +)$ ([1], [2], [15]). Using (iii) further we have $L_c R_d\theta = L_c L_d I_d\theta = L_a\beta$. □

We denote by $Z(Q, +)$ the centre of a group $(Q, +)$, i.e. $Z(Q, +) = \{a \in Q \,|\, a + x = x + a \quad \forall x \in Q\}$.

Lemma 2. *Let (Q, g) be an n-ary quasigroup of the form $g(x_1^n) = x_1 + x_2 + \cdots + x_n$ where $(Q, +)$ is a group. A permutation θ of the set Q is an automorphism of the quasigroup (Q, g) if and only if $\theta = L_k\varphi$ where $\varphi \in Aut(Q, +)$, $k \in Z(Q, +)$, $(n-1)k = 0$.*

Proof. Let $\theta \in Aut(Q, g)$. Then

$$\theta(x_1 + x_2 + \cdots + x_n) = \theta x_1 + \theta x_2 + \cdots + \theta x_n \tag{3}$$

for all $x_i \in Q, i \in \{1, 2, \ldots, n\}$. If we take in (3) $x_1 = x_2 = \cdots = x_n = 0$, then we have $n\theta 0 = \theta 0$, $(n-1)\theta 0 = 0$.

If we take in (3) $x_2 = x_3 = \cdots = x_n = 0$, then $\theta x_1 = \theta x_1 + \theta 0 + \cdots + \theta 0 = R_{(n-1)\theta 0}\theta x_1$. If we substitute in (3) $x_1 = x_3 = \cdots = x_n = 0$, then $\theta x_2 = \theta 0 + \theta x_2 + \theta 0 + \cdots + \theta 0 = L_{\theta 0}R_{(n-2)\theta 0}\theta x_2$. If in the last two equalities we rename x_1 and x_2 by x and compare the right sides of these equalities, then we obtain $\theta x + \theta 0 + \cdots + \theta 0 = \theta 0 + \theta x + \theta 0 + \cdots + \theta 0$, $\theta x + \theta 0 = \theta 0 + \theta x$. Thus $\theta 0 \in Z(Q, +)$.

If we take in (3) $x_3 = \cdots = x_n = 0$, then $\theta(x_1 + x_2) = \theta x_1 + R_{(n-2)\theta 0}\theta x_2$. Therefore the permutation θ is a quasiautomorphism of group $(Q, +)$. Moreover, $\theta = L_{\theta 0}\varphi$ where $\varphi \in Aut(Q, +)$. Indeed, any group quasiautomorphism has form $L_a\varphi$. But $a = \theta 0$ because $L_a\varphi 0 = L_a 0 = a = \theta 0$.

Therefore we obtain that $\theta = L_{\theta 0}\varphi$, where $(n-1)\theta 0 = 0$, $\theta 0 \in Z(Q, +)$ and $\varphi \in Aut(Q, +)$.

Converse. Let $\theta = L_k\varphi$ where $\varphi \in Aut(Q, +)$, $k \in Z(Q, +)$, $(n-1)k = 0$. Let us prove that $\theta \in Aut(Q, g)$. We have $\theta(x_1 + x_2 + \cdots + x_n) = L_k\varphi(x_1 + x_2 + \cdots + x_n) = k + \varphi x_1 + \varphi x_2 + \cdots + \varphi x_n = nk + \varphi x_1 + \varphi x_2 + \cdots + \varphi x_n = k + \varphi x_1 + k + \varphi x_2 + \cdots + k + \varphi x_n = \theta x_1 + \theta x_2 + \cdots + \theta x_n$. $\square$

Lemma 3. *Let (Q, g) be an n-ary quasigroup of the form $g(x_1^n) = x_1 + x_2 + \cdots + x_n$, where $(Q, +)$ is a group. An $(n+1)$-tuple $T = (\alpha_1, \ldots, \alpha_n, \gamma)$ of permutations of the set Q is an autotopy of the quasigroup (Q, g) if and only if*

$$\begin{array}{l} T = (L_{a_1}I_{a_1}, L_{a_2}I_{a_1+a_2}, L_{a_3}I_{a_1+a_2+a_3}, \ldots, L_{a_n}I_t, R_t)\circ \\ (\varphi, \varphi, \varphi, \ldots, \varphi), \end{array} \tag{$\star$}$$

where $t = a_1 + a_2 + a_3 + \cdots + a_n$, $\varphi \in Aut(Q, +)$, $a_1^n \in Q$.

Proof. Let $(\alpha_1, \ldots, \alpha_n, \gamma)$ be an autotopy of a quasigroup (Q, g). Then

$$\gamma(x_1 + \cdots + x_n) = \alpha_1 x_1 + \cdots + \alpha_n x_n. \tag{4}$$

If in (4) all $x_i = 0$, then $\gamma 0 = \sum_{i=1}^{n} \alpha_i 0$. If in (4) only $x_i \neq 0$ for any fixed value i, then $\gamma x_i = L_{\alpha_1 0 + \cdots + \alpha_{i-1} 0} R_{\alpha_{i+1} 0 + \cdots + \alpha_n 0}\alpha_i x_i$. Further, if we take into consideration Lemma 1 (ii), we have

$$\alpha_i = R_{-(\alpha_{i+1}0+\dots+\alpha_n 0)} L_{-(\alpha_1 0+\dots+\alpha_{i-1}0)}\gamma = \\ L_{-(\alpha_1 0+\dots+\alpha_{i-1}0)} R_{-(\alpha_{i+1}0+\dots+\alpha_n 0)}\gamma.$$

If in equality (4) we re-write all permutations α_i in their last form, then we have

$$\gamma(x_1 + \dots + x_n) = \\ \gamma x_1 - \alpha_n 0 - \dots - \alpha_1 0 + \gamma x_2 - \alpha_n 0 - \dots - \alpha_1 0 + \gamma x_3 - \dots + \gamma x_n. \tag{5}$$

Let $d = (\alpha_1 0+\dots+\alpha_n 0)$. Then from equality (5) we have $\gamma(x_1+\dots+x_n) - d = \gamma x_1 - d + \gamma x_2 - d + \dots + \gamma x_n - d$. Therefore $R_{-(\alpha_1 0+\dots+\alpha_n 0)}\gamma = R_{-d}\gamma \in Aut(Q, g)$.

By Lemma 2 any automorphism of quasigroup (Q, g) has the form $L_k\varphi$, where $k \in Z(Q, +)$, $\varphi \in Aut(Q, +)$. Then $R_{-d}\gamma = L_k\varphi = R_k\varphi$, $\gamma = R_d R_k\varphi = R_{k+d}\varphi$.

We remark that, if $I_b x = -b + x + b$ is the inner automorphism of the group $(Q, +)$, then $I_b = L_{-b}R_b$ or $R_b = L_b I_b$. Further we have

$$\alpha_i = L_{-(\alpha_1 0+\dots+\alpha_{i-1}0)} R_{-(\alpha_{i+1}0+\dots+\alpha_n 0)}\gamma = \\ L_{-(\alpha_1 0+\dots+\alpha_{i-1}0)} R_{-\alpha_n 0-\dots-\alpha_{i+1}0} R_{\alpha_1 0+\dots+\alpha_n 0} R_k\varphi = \\ L_{-(\alpha_1 0+\dots+\alpha_{i-1}0)} R_{\alpha_1 0+\dots+\alpha_i 0} R_k\varphi = \\ L_{\alpha_i 0} L_{-\alpha_i 0} L_{-(\alpha_1 0+\dots+\alpha_{i-1}0)} R_{\alpha_1 0+\dots+\alpha_i 0} R_k\varphi = \\ L_{\alpha_i 0} L_{-(\alpha_1 0+\dots+\alpha_i 0)} R_{\alpha_1 0+\dots+\alpha_i 0} R_k\varphi = \\ L_{\alpha_i 0} I_{\alpha_1 0+\dots+\alpha_i 0} R_k\varphi = L_{\alpha_i 0} R_k I_{\alpha_1 0+\dots+\alpha_i 0}\varphi.$$

We have used that if $k \in Z(Q, +)$, then $R_k I_b = I_b R_k$. Indeed, $-b + x + b + k = -b + x + k + b = I_b R_k x$. Further we obtain $\alpha_i = L_{\alpha_i 0+k} I_{\alpha_1 0+\dots+\alpha_i 0}\varphi$, since for every element k from the centre of the group $(Q, +)$ $R_k = L_k$.

We denote $\alpha_i 0 + k$ as a_i and $d + k$ as t. Let us prove that $\sum_{i=1}^{n} a_i = t$. Indeed, we have $a_1 + a_2 + \dots + a_n = \alpha_1 0 + k + \alpha_2 0 + k + \dots + \alpha_n 0 + k = \alpha_1 0 + \alpha_2 0 + \dots + \alpha_n 0 + k = d + k = t$. Therefore we obtain that $(n+1)$-tuple

$$T = (L_{a_1} I_{a_1}\varphi, L_{a_2} I_{a_1+a_2}\varphi, L_{a_3} I_{a_1+a_2+a_3}\varphi, \dots, L_{a_n} I_t\varphi, R_t\varphi)$$

is an autotopy of the quasigroup (Q, g).

Converse. We shall prove that any $(n + 1)$-tuple of such form is an autotopy of the quasigroup (Q, g). Let $n = 3$. We have $a_1 - a_1 + \varphi x_1 + a_1 + a_2 - a_2 - a_1 + \varphi x_2 + a_1 + a_2 + a_3 - a_3 - a_2 - a_1 + \varphi x_3 + a_1 + a_2 + a_3 = R_t\varphi(x_1 + x_2 + x_3)$. Further past cancellation in the left-hand side of the last relation we obtain $\varphi x_1 + \varphi x_3 + \varphi x_3 + a_1 + a_2 + a_3 = R_t\varphi(x_1 + x_2 + x_3)$. For other values of arity n the proof is similar. □

Corollary 1. *Any autotopy of the quasigroup (Q, g) with the form $g(x_1^n) = \Sigma_{i=1}^n(x_i)$ over a group $(Q, +)$ has unique representation in the form $(\star)$.*

Proof. If we suppose that

$$\begin{array}{l}(L_{a_1}I_{a_1}\varphi, L_{a_2}I_{a_1+a_2}\varphi, L_{a_3}I_{a_1+a_2+a_3}\varphi, \dots, L_{a_n}I_t\varphi, R_t\varphi) = \\ (L_{b_1}I_{b_1}\psi, L_{b_2}I_{b_1+b_2}\psi, L_{b_3}I_{b_1+b_2+b_3}\psi, \dots, L_{b_n}I_d\psi, R_d\psi),\end{array}$$

then we have $L_{a_1}I_{a_1}\varphi = L_{b_1}I_{b_1}\psi$, $L_{a_1}I_{a_1}\varphi\, 0 = L_{b_1}I_{b_1}\psi\, 0$, $a_1 = b_1$, $\varphi = \psi$. Further we obtain $L_{a_2}I_{a_1+a_2} = L_{b_2}I_{b_1+b_2}$, $L_{a_2}I_{a_1+a_2}0 = L_{b_2}I_{b_1+b_2}0$, $a_2 = b_2$ and so on. Thus $a_i = b_i$ for all $i \in \overline{1, n}$, $t = d$, $\varphi = \psi$. □

Remark 2. The change of distribution of brackets in equality (5) permits us to obtain from one autotopy of the quasigroup (Q, g) other autotopies of this quasigroup.

Proposition 1. *Let (Q, g) be a finite n-ary quasigroup of order $|Q|$ with the form $g(x_1^n) = x_1 + x_2 + \dots + x_n$, where $(Q, +)$ is a group. Then*

$$|\mathfrak{T}(Q, g)| = |Q|^n \cdot |Aut(Q, +)|.$$

Proof. From Lemma 3 it follows that any autotopy T of the quasigroup (Q, g) has the form $T = T_1 \circ T_2$, where

$$\begin{array}{l}T_1 = (L_{a_1}I_{a_1}, L_{a_2}I_{a_1+a_2}, L_{a_3}I_{a_1+a_2+a_3}, \dots, L_{a_n}I_t, R_t), \\ T_2 = (\varphi, \varphi, \varphi, \dots, \varphi), \\ t = a_1 + a_2 + a_3 + \dots + a_n, \varphi \in Aut(Q, +), a_1^n \in Q.\end{array}$$

Let

$$\begin{array}{l}\mathfrak{T}_1 = \{(L_{a_1}I_{a_1}, L_{a_2}I_{a_1+a_2}, L_{a_3}I_{a_1+a_2+a_3}, \dots, L_{a_n}I_t, R_t) \mid \\ \forall\, a_1^n \in Q\}, \mathfrak{T}_2 = \{(\varphi, \varphi, \varphi, \dots, \varphi) \mid \varphi \in Aut(Q, +)\}.\end{array}$$

Taking into consideration Corollary 1 it is easy to see that $|\mathfrak{T}_1| = |Q|^n$. It is clear that $|\mathfrak{T}_2| = |Aut(Q, +)|$.

We prove that $\mathfrak{T}_1 \cap \mathfrak{T}_2 = (\varepsilon, \varepsilon, \dots, \varepsilon)$. Indeed, if $L_{a_1}I_{a_1} = \varphi$, then $L_{a_1}I_{a_1}0 = \varphi 0$, $a_1 = 0$, $\varphi = \varepsilon$.

From Lemma 3 it follows that any n-tuple $T_1 \in \mathfrak{T}_1$ and any n-tuple $T_2 \in \mathfrak{T}_2$ is an autotopy of the quasigroup (Q, g). Therefore $|\mathfrak{T}(Q, g)| = |Q|^n \cdot |Aut(Q, +)|$. □

Corollary 2. *Let (Q, g) be a finite n-ary quasigroup of order $|Q|$ with the form $g(x_1^n) = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n$, where $(Q, +)$ is a group, α_1^n are permutations of the set Q. Then*

$$|\mathfrak{T}(Q, g)| = |Q|^n \cdot |Aut(Q, +)|.$$

Proof. This follows from Proposition 1 and Theorem 1 since finite isomorphic groups have equal orders. □

Corollary 3. *Let (Q, g) be an n-ary quasigroup of the form $g(x_1^n) = x_1 + x_2 + \cdots + x_n$, where $(Q, +)$ is an abelian group. An $(n+1)$-tuple $T = (\alpha_1, \ldots, \alpha_n, \gamma)$ of permutations of the set Q is an autotopy of the quasigroup (Q, g) if and only if $T = (L_{a_1}\varphi, L_{a_2}\varphi, L_{a_3}\varphi, \ldots, L_{a_n}\varphi, L_t\varphi)$, where $t = a_1 + a_2 + a_3 + \cdots + a_n$, $\varphi \in Aut(Q, +)$, $a_1^n \in Q$.*

Proof. In abelian group $I_a = \varepsilon$, $L_a = R_a$ for all $a \in Q$. □

Theorem 3. *If an n-ary quasigroup (Q, g) has the form $g(x_1^n) = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$ where $(Q, +)$ is an abelian group, α_1^n are permutations of the set Q, then*

$$\mathfrak{T}(Q, g) \cong \bigoplus_{i=1}^{n} (Q, +)_i \leftthreetimes Aut(Q, +),$$

where $\bigoplus_{i=1}^{n} (Q, +)_i$ is a direct sum of n copies of the group $(Q, +)$.

Proof. By Theorem 1, autotopy groups of isotopic n-ary quasigroups are isomorphic (moreover, autotopy groups of isostrophic n-ary quasigroups are isomorphic [3]). Therefore it is sufficient to prove this theorem for an n-ary quasigroup (Q, f) with the form $f(x_1^n) = x_1 + x_2 + \cdots + x_n$ where $(Q, +)$ is an abelian group.

From Corollary 3 it follows that any autotopy T of the quasigroup (Q, g) has the form $T = T_1 \circ T_2$, where

$$\begin{aligned} &T_1 = (L_{a_1}, L_{a_2}, L_{a_3}, \ldots, L_{a_n}, L_t), \\ &T_2 = (\varphi, \varphi, \varphi, \ldots, \varphi), \\ &t = a_1 + a_2 + a_3 + \cdots + a_n, \varphi \in Aut(Q, +), a_1^n \in Q. \end{aligned}$$

Let

$$\begin{aligned} &\mathfrak{T}_1 = \{(L_{a_1}, L_{a_2}, L_{a_3}, \ldots, L_{a_n}, L_t) \mid t = a_1 + \ldots a_n, \text{ for all } \mathrm{a_1^n} \in \mathrm{Q}\}, \\ &\mathfrak{T}_2 = \{(\varphi, \varphi, \varphi, \ldots, \varphi) \mid \text{for all } \varphi \in \mathrm{Aut(Q, +)}\}. \end{aligned}$$

From Corollary 3 it follows that $\mathfrak{T}(Q, g) = \mathfrak{T}_1 \circ \mathfrak{T}_2$.

We shall prove that $\mathfrak{T}_1 \cap \mathfrak{T}_2 = (\varepsilon, \varepsilon, \ldots, \varepsilon)$. Indeed, if $L_{a_1} = \varphi$, then $L_{a_1}0 = \varphi 0$, $a_1 = 0$, $\varphi = \varepsilon$.

From Lemma 3 it follows that any n-tuple $T_1 \in \mathfrak{T}_1$ and any n-tuple $T_2 \in \mathfrak{T}_2$ is an autotopy of the quasigroup (Q, g).

The set $\mathfrak{T}_1$ forms a group with respect to the term by term multiplication of $(n+1)$-tuples of the set $\mathfrak{T}_1$. Really, let

$$\begin{aligned} &T_1 = (L_{a_1}, L_{a_2}, \ldots, L_{a_n}, L_d), \\ &T_2 = (L_{b_1}, L_{b_2}, \ldots, L_{b_n}, L_t), \end{aligned}$$

$T_1, T_2 \in \mathfrak{T}_1$. We prove that $T_1 \circ T_2 \in \mathfrak{T}_1$.

We have $L_{a_1}L_{b_1} = L_{a_1+b_1}, \dots, L_{a_n}L_{b_n} = L_{a_n+b_n}, L_dL_t = L_{d+t}, a_1 + b_1 + \dots + a_n + b_n = d + t$. Then $T_1 \circ T_2 \in \mathfrak{T}_1$. From Lemma 1 (ii) it follows that $T_1^{-1} = (L_{-a_1}, L_{-a_2}, \dots, L_{-a_n}, L_{-d})$. Thus $T_1^{-1} \in \mathfrak{T}_1$.

Therefore $(\mathfrak{T}_1, \circ)$ is a group and it is easy to see that $(\mathfrak{T}_1, \circ) \cong \bigoplus_{i=1}^{n}(Q, +)_i$.

It is clear that $(\mathfrak{T}_2, \circ) \cong Aut(Q, +)$, $(\mathfrak{T}_1, \circ) \subseteq \mathfrak{T}(Q, g)$, $(\mathfrak{T}_2, \circ) \subseteq \mathfrak{T}(Q, g)$.

We shall prove that $(\mathfrak{T}_1, \circ) \trianglelefteq \mathfrak{T}(Q, g)$. Let $T = (L_{a_1}\varphi, L_{a_2}\varphi, L_{a_3}\varphi, \dots, L_{a_n}\varphi, L_t\varphi) \in \mathfrak{T}$, $T_1 = (L_{b_1}, L_{b_2}, L_{b_3}, \dots, L_{b_n}, L_d) \in \mathfrak{T}_1$.

From Lemma 1 it follows that

$$\begin{aligned} T^{-1} = \ & (L_{-\varphi^{-1}a_1}\varphi^{-1}, L_{-\varphi^{-1}a_2}\varphi^{-1}, L_{-\varphi^{-1}a_3}\varphi^{-1}, \dots, \\ & L_{-\varphi^{-1}a_n}\varphi^{-1}, L_{-\varphi^{-1}t}\varphi^{-1}). \end{aligned}$$

Further we have

$$\begin{aligned} T^{-1} \circ T_1 \circ T = \ & (L_{-\varphi^{-1}a_1}\varphi^{-1}L_{b_1}L_{a_1}\varphi, L_{-\varphi^{-1}a_2}\varphi^{-1}L_{b_2}L_{a_2}\varphi, \dots, \\ & L_{-\varphi^{-1}a_n}\varphi^{-1}L_{b_n}L_{a_n}\varphi, L_{-\varphi^{-1}t}\varphi^{-1}L_dL_t\varphi) = \\ & (L_{-\varphi^{-1}a_1+\varphi^{-1}b_1+\varphi^{-1}a_1}, L_{-\varphi^{-1}a_2+\varphi^{-1}b_2+\varphi^{-1}a_2}, \dots, \\ & L_{-\varphi^{-1}a_n+\varphi^{-1}b_n+\varphi^{-1}a_n}, L_{-\varphi^{-1}t+\varphi^{-1}d+\varphi^{-1}t}) \in \mathfrak{T}_1 \end{aligned}$$

By proving the last equality we have used Lemma 1.

Therefore $\mathfrak{T}(Q, g) \cong \bigoplus_{i=1}^{n}(Q, +)_i \leftthreetimes Aut(Q, +)$. □

Let $Z^{(n-1)}(Q, +) = \{a \in Z(Q, +) \,|\, (n-1)a = 0\}$, where $(Q, +)$ is a group and $Z(Q, +)$ is the centre of this group. It is easy to see that $Z^{(n-1)}(Q, +)$ is a subgroup of the group $(Q, +)$. Let $\overline{Z}(Q, +)$ be a group that consists from all left translations of the group $(Q, +)$ such that $x \in Z^{(n-1)}(Q, +)$, i.e. $\overline{Z}(Q, +) \subseteq M(Q, +)$.

Proposition 2. *In a quasigroup (Q, g) with the form $g(x_1^n) = x_1 + x_2 + \dots + x_n$, where $(Q, +)$ is a binary group,*

$$Aut(Q, g) \cong \overline{Z}(Q, +) \leftthreetimes Aut(Q, +).$$

Proof. Really, $\overline{Z}(Q, +) \bigcap Aut(Q, +) = \varepsilon$, $\overline{Z}(Q, +) \triangleleft Aut(Q, g)$, and, further, we have

$$\begin{aligned} & \overline{Z}(Q, +) \cdot Aut(Q, +) = Aut(Q, g), \\ & Aut(Q, g)/\overline{Z}(Q, +) \cong Aut(Q, +), \\ & Aut(Q, g) \cong \overline{Z}(Q, +) \leftthreetimes Aut(Q, +). \end{aligned}$$

□

Corollary 4. *In a quasigroup (Q, g) with the form $g(x_1^n) = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n$, where $(Q, +)$ is a binary group, α_1^n are permutations of the set Q,*

$$Aut(Q, g) \cong H \subseteq (Q, +) \leftthreetimes Aut(Q, +).$$

Proof. From definition of isotopy it follows that $(Q, g) = (Q, f)(\alpha_1, \alpha_2, \dots, \alpha_n, \varepsilon)$, where the n-ary quasigroup (Q, f) has the form $f(x_1^n) = x_1 + x_2 + \dots + x_n$ over the group $(Q, +)$.

Then from Theorem 1 it follows that any quasiautomorphism of the n-ary quasigroup (Q, g) has the same form as the form of any quasiautomorphism of the n-ary quasigroup (Q, f).

Thus from Lemma 3 and Lemma 1 (iii) it follows that any quasiautomorphism of the n-ary quasigroup (Q, g) has the form $L_a^+\varphi$, where $\varphi \in Aut(Q, +)$. Therefore any automorphism of the quasigroup (Q, g) has the same form.

From Lemma 3 it follows that any permutation of the form $L_a^+\varphi$ is an quasiautomorphism of the n-ary quasigroups (Q, g) and (Q, f).

It is easy to check that set of all quasiautomorphisms of the quasigroup (Q, g) forms a group with respect to usual operation multiplication of permutations of the set Q. It is well known ([10]) that this group is isomorphic to the group $(Q, +) \leftthreetimes Aut(Q, +)$.

Therefore $Aut(Q, g) \cong H \subseteq (Q, +) \leftthreetimes Aut(Q, +)$. □

Remark 3. Binary analog of Corollary 4 was proved in [18].

3. On automorphisms of n-ary T-quasigroups

In this section we study automorphisms and automorphism groups of some classes of n-ary T-quasigroups. We prove that every finite medial n-ary quasigroup of order greater than 2 has a non-identity automorphism group. We recall an example of an infinite medial quasigroup with identity automorphism group [22].

The binary analog of Proposition 3 was proved in [18], n-ary analog of Proposition 3 was claimed in [21].

It is known that centralizer of elements $\varphi_1, \dots, \varphi_n$ in group $Aut(Q, +)$ is the following set $\{\omega \in Aut(Q, +) \mid \omega\varphi_i = \varphi_i\omega \, \forall\, i \in \overline{1, n}\}$ ([10]). Denote this set as $C_{Aut(Q,+)}(\varphi_1, \dots, \varphi_n)$. Sometimes we shall denote this set only by letter C. The set C forms a group with respect to usual operation of multiplication of permutations of the set Q ([10]).

Proposition 3. *A permutation γ of a set Q is an automorphism of n-T-quasigroup (Q, f) of the form $f(x_1, x_2, \dots, x_n) = \varphi_1 x_1 + \varphi_2 x_2 +$*

$\cdots + \varphi_n x_n + a$ *if and only if* $\gamma = L_b^+\beta$, *where* $\beta a - a = \delta b$, $\beta \in C_{Aut(Q,+)}(\varphi_1, \dots, \varphi_n)$, $\delta = \sum_{i=1}^{n} \varphi_i - \varepsilon$, $b \in Q$.

Proof. From Corollary 3 and Theorem 1 it follows that any autotopy T of the n-T-quasigroup (Q, f) has the following form:

$$\begin{aligned} T = \ & (L_{\varphi_1^{-1}(a_1)}\varphi_1^{-1}\psi\varphi_1, L_{\varphi_2^{-1}(a_2)}\varphi_2^{-1}\psi\varphi_2, \dots, \\ & L_{\varphi_n^{-1}(a_n)}\varphi_n^{-1}\psi\varphi_n, L_t\psi), \end{aligned}$$

where ψ is an automorphism of abelian group $(Q, +)$, all translations in the last equality are translations of the abelian group $(Q, +)$. It is so since $\varphi^{-1}L_x\psi\varphi = L_{\varphi^{-1}x}\varphi^{-1}\psi\varphi$ for all $x \in Q$.

Since any automorphism of n-T-quasigroup (Q, f) is an autotopy with equal components, then we can write any automorphism of quasigroup (Q, f) in the form $L_b\beta$ where $\beta \in Aut(Q, +)$ and L_b is a left translation of the group $(Q, +)$.

Such presentation of this quasigroup automorphism is unique. Indeed, if $L_d\rho = L_b\beta$, then $L_d\rho 0 = L_b\beta 0$, $d = b$, $\rho = \beta$.

Let $L_b\beta \in Aut(Q, f)$, i.e. $\varphi_1 L_b\beta x_1 + \varphi_2 L_b\beta x_2 + \cdots + \varphi_n L_b\beta x_n + a = L_b\beta(\varphi_1 x_1 + \varphi_2 x_2 + \cdots + \varphi_n x_n)$. Since $\varphi_i \in Aut(Q, +)$ for any $i \in \overline{1, n}$, we have

$$\begin{aligned} & \varphi_1 b + \varphi_1\beta x_1 + \varphi_2 b + \varphi_2\beta x_2 + \cdots + \varphi_n b + \varphi_n\beta x_n + a = \\ & b + \beta\varphi_1 x_1 + \beta\varphi_2 x_2 + \cdots + \beta\varphi_n x_n + \beta a. \end{aligned} \tag{6}$$

If we take in (6) $x_i = 0$ for all i, then we have

$$\varphi_1 b + \varphi_2 b + \cdots + \varphi_n b + a = b + \beta a. \tag{7}$$

If we denote $\sum_{i=1}^{n} \varphi_i - \varepsilon$ by δ, then we have $\beta a - a = \delta b$.

If we take into consideration equality (7), then from (6) it follows

$$\varphi_1\beta x_1 + \varphi_2\beta x_2 + \cdots + \varphi_n\beta x_n = \beta\varphi_1 x_1 + \beta\varphi_2 x_2 + \cdots + \beta\varphi_n x_n. \tag{8}$$

If we take in (8) $x_2 = x_3 = \cdots = x_n = 0$, then we have $\varphi_1\beta = \beta\varphi_1$. Similarly we obtain that $\varphi_2\beta = \beta\varphi_2, \dots, \varphi_n\beta = \beta\varphi_n$, i.e. $\beta \in C_{Aut(Q,+)}(\varphi_1, \dots, \varphi_n) = C$.

It is easy to check up that the converse is correct too. Indeed,

$$\begin{aligned} & \gamma(f(x_1^n)) = \gamma(\textstyle\sum_{i=1}^{n}(\varphi_i x_i) + a) = L_b\beta(\sum_{i=1}^{n}(\varphi_i x_i) + a) = \\ & b + \textstyle\sum_{i=1}^{n} \beta\varphi_i x_i + \beta a = \\ & \textstyle\sum_{i=1}^{n}(\varphi_i b) + \sum_{i=1}^{n}(\varphi_i\beta x_i) + a = \\ & \textstyle\sum_{i=1}^{n}(\varphi_i(L_b\beta x_i)) + a = f(\gamma x_1, \dots, \gamma x_n). \end{aligned}$$

□

Corollary 5. *If $L_b\beta \in Aut(Q, f)$, where (Q, f) is an n-T-quasigroup with the form $f(x_1^n) = \varphi_1 x_1 + \varphi_2 x_2 + \dots + \varphi_n x_n$, then $L_b \in Aut(Q, f)$ and $\beta \in Aut(Q, f)$.*

Proof. If $L_b\beta \in Aut(Q, f)$, then equality (7) is fulfilled. Equality (7) in case $a = 0$ has the form $b = \sum_{i=1}^{n}(\varphi_i b)$. Further we have

$$\begin{array}{l} L_b f(x_1^n) = L_b(\sum_{i=1}^{n}(\varphi_i x_i)) = \\ b + \sum_{i=1}^{n} \varphi_i x_i = \sum_{i=1}^{n}(\varphi_i b) + \sum_{i=1}^{n} \varphi_i x_i = \\ \sum_{i=1}^{n} \varphi_i(b + x_i) = f(L_b x_1, \dots, L_b x_n). \end{array}$$

Thus $L_b \in Aut(Q, f)$. If $L_b\beta \in Aut(Q, f)$, $L_b \in Aut(Q, f)$, then $\beta \in Aut(Q, f)$. □

Proposition 4. *If an n-T-quasigroup (Q, g) has the form $g(x_1^n) = \varphi_1 x_1 + \varphi_2 x_2 + \dots + \varphi_n x_n$, then*

$$Aut(Q, g) \cong K \rtimes C,$$

where $K = \{L_b^+ \mid b \in Q, \varphi_1 b + \varphi_2 b + \dots + \varphi_n b = b\}$, $C = \{\omega \in Aut(Q, +) \mid \omega\varphi_i = \varphi_i\omega \, \forall \, i \in \overline{1, n}\}$.

Proof. From Proposition 3 and Corollary 5 it follows that sets K and C are subgroups of group $Aut(Q, g)$ and that $Aut(Q, g) = K \cdot C$. Since $K \cap C = \varepsilon$, $K \trianglelefteq Aut(Q, g)$ we have $Aut(Q, g) \cong K \rtimes C$. □

An element d of an n-ary quasigroup (Q, f) such that $f(d, \dots, d) = d$ is called an *idempotent element* of quasigroup (Q, f) (in brackets we have taken the element d exactly n times). In binary case an element $d \in Q$ such that $d \cdot d = d$ is called an idempotent element of quasigroup $(Q, \cdot)$.

Corollary 6. *If n-T-quasigroup (Q, g) with the form $g(x_1, \dots, x_n) = \sum_{i=1}^{n} \varphi_i x_i$ has exactly one idempotent element, then*

$$Aut(Q, g) \cong C,$$

where $C = \{\omega \in Aut(Q, +) \mid \omega\varphi_i = \varphi_i\omega \, \forall \, i \in \overline{1, n}\}$.

Proof. In this case $K = \{\varepsilon\}$. □

Corollary 7. *If n-T-quasigroup (Q, g) is an idempotent quasigroup with the form $g(x_1, \dots, x_n) = \sum_{i=1}^{n} \varphi_i x_i$ over an abelian group $(Q, +)$, then*

$$Aut(Q, g) \cong (Q, +) \rtimes C,$$

$C = \{\omega \in Aut(Q, +) \mid \omega\varphi_i = \varphi_i\omega \, \forall \, i \in \overline{1, n}\}$.

Proof. We have $K \cong (Q, +)$ in case when (Q, g) is an idempotent quasigroup. □

Lemma 4. *An n-T-quasigroup (Q, f) $f(x_1, \dots, x_n) = \sum_{i=1}^{n} \varphi_i x_i + a$ has at least one idempotent element if and only if it is isomorphic to n-T-quasigroup (Q, g) with the form $g(x_1^n) = \sum_{i=1}^{n} \varphi_i x_i$ over the same abelian group $(Q, +)$.*

Proof. Let element u be an idempotent element of n-ary T-quasigroup (Q, f) with the form $f(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} \varphi_i x_i + a$. If we take in the last equality $x_1 = x_2 = \dots = x_n = u$, then we have $\sum_{i=1}^{n} \varphi_i u + a = u$, i.e.

$$\sum_{i=1}^{n} \varphi_i u + a - u = 0. \tag{9}$$

Then isomorphic image of the n-T-quasigroup (Q, f) with an isomorphism T of form $(L_u, \cdots, L_u)$ ((n+1) times), where L_u is a left translation of the group $(Q, +)$, will be n-T-quasigroup (Q, g) with form $g(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} \varphi_i x_i$. Really, we have

$$\begin{aligned}
&L_{-u} f(L_u x_1, \cdots, L_u x_n) = \\
&-u + \varphi_1 u + \varphi_1 x_1 + \varphi_2 u + \varphi_2 x_2 + \cdots + \varphi_n u + \varphi x_n + a = \\
&-u + \varphi_1 u + \varphi_2 u + \cdots + \varphi_n u + a + \textstyle\sum_{i=1}^{n} \varphi_i x_i = \\
&\textstyle\sum_{i=1}^{n} \varphi_i u + a - u + \sum_{i=1}^{n} \varphi_i x_i = \\
&0 + \textstyle\sum_{i=1}^{n} \varphi_i x_i = g(x_1, \cdots, x_n).
\end{aligned}$$

Any n-T-quasigroup (Q, g) with the form $\sum_{i=1}^{n} \varphi_i x_i$ has at least one idempotent element. Indeed, $\sum_{i=1}^{n} \varphi_i 0 = 0$. Then the n-T-quasigroup (Q, f), that is an isomorphic copy of n-T-quasigroup (Q, g), has an idempotent element. The lemma is proved. □

Condition (9) that the n-T-quasigroup (Q, f) with the form $f(x_1^n) = \sum_{i=1}^{n} \varphi_i x_i + a$ has an idempotent element u we can re-write in the form $\delta u = -a$, where δ is an endomorphism of abelian group $(Q, +)$ such that $\delta = \sum_{i=1}^{n} \varphi_i - \varepsilon$.

Hence we can formulate the following conditions when the n-T-quasigroup (Q, f) has an idempotent element.

Lemma 5. *An n-T-quasigroup (Q, f), $f(x_1, \dots, x_n) = \sum_{i=1}^{n} \varphi_i x_i + a$, has at least one idempotent element if and only if there exists element $d \in Q$ such that $\delta d = -a$.*

Remark 4. An n-T-quasigroup (Q, f), $f(x_1, \dots, x_n) = \sum_{i=1}^{n} \varphi_i x_i + a$, has exactly one idempotent element if and only if the endomorphism δ is a permutation of the set Q, i.e. if and only if the endomorphism δ is an automorphism of the quasigroup (Q, f).

For binary case Lemmas 4 and 5 are easy received from the remark on page 109 in [18].

Lemma 6. *T-quasigroup* $(Q, \cdot)$, $x \cdot y = \varphi_1 x + \varphi_2 y + a$, *has at least one idempotent element if and only if it is isomorphic to T-quasigroup* $(Q, \circ)$ *with the form* $x \circ y = \varphi_1 x + \varphi_2 y$ *over the same abelian group* $(Q, +)$.

Lemma 7. *T-quasigroup* $(Q, \cdot)$ *with the form* $x \cdot y = \varphi_1 x_1 + \varphi_2 y + a$ *has at least one idempotent element if and only if there exists element* $d \in Q$ *such that* $\delta d = -a$.

Theorem 4. *If n-T-quasigroup* (Q, g) *with the form* $g(x_1, \dots, x_n) = \sum_{i=1}^{n} \varphi_i x_i + a$ *has at least one idempotent element, then*

$$Aut(Q, g) \cong K \leftthreetimes C,$$

where $K = \{L_b^+ \mid b \in Q, \varphi_1 b + \varphi_2 b + \cdots + \varphi_n b = b\}$, $C = \{\omega \in Aut(Q, +) \mid \omega\varphi_i = \varphi_i \omega \; \forall i \in \overline{1, n}\}$.

Proof. It is sufficient to take into consideration Lemma 4 and Proposition 4. □

Remark 5. In binary case from idempotency of T-quasigroup $(Q, \cdot)$ it follows that this quasigroup is medial and distributive. Really if $x \cdot x = \varphi x + \psi x = x$ for any $x \in Q$, then $\varphi + \psi = \varepsilon$, $\varphi = -\psi + \varepsilon$ and then $\varphi\psi = (-\psi + \varepsilon)\psi = -\psi^2 + \psi = \psi(-\psi + \varepsilon) = \psi\varphi$.

We notice that even in ternary case there exist non-medial idempotent T-quasigroups.

Example 1. Let $(Q, +) = (Z_5 \oplus Z_5, +)$ be a direct sum of two cyclic groups of order 5. Let

$$\alpha_1 = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 3 & 0 \\ 3 & 3 \end{pmatrix},$$

where, for example,

$$\begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 \cdot y_1 + 2 \cdot y_2 \\ 2 \cdot y_1 + 1 \cdot y_2 \end{pmatrix}$$

for all $(y_1, y_2) \in Z_5 \oplus Z_5$.

A quasigroup (Q, f) of the form $f(x_1, x_2, x_3) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$ for all $x_1, x_2, x_3 \in Q$ is an idempotent non-medial ($\alpha_1\alpha_2 \neq \alpha_2\alpha_1$) 3-ary T-quasigroup.

Remark 6. It is clear that with growing of arity of a quasigroup operation an influence of idempotent elements on properties of a quasigroup will be weakened. Really, in binary idempotent quasigroup (Q, f_2) we have $|Q|^2$ quasigroup "words" of the form $(x, y, f_2(x, y))$ and $|Q|$ idempotent elements, in ternary quasigroup (Q, f_3) we shall have $|Q|^3$ such quasigroup words and $|Q|$ idempotent elements and so on.

Example 2. Medial quasigroup $(Z, \circ)$ over infinite cyclic group $(Z, +)$ of the form $x \circ y = -x - y + 1$ has identity automorphism group ([22]).

Proof. From Proposition 3 it follows that any automorphism of the quasigroup $(Z, \star)$, $x \star y = -x - y + a$ has form $L_b^+ \psi$, where $\psi \in Aut(Z, +) \cong (Z_2, \cdot)$ ([10]). We suppose that $Z_2 = \{1, -1\}$.

Then we have $L_b^+\psi(x \circ y) = L_b^+\psi x \circ L_b^+\psi y$, $b - \psi x - \psi y + \psi a = -b - \psi x - b - \psi y + a$, $\psi a - a = -3b$. If $\psi = 1$, then $0 = -3b$, $b = 0$ and we obtain that $L_b^+\psi = \varepsilon$ is an automorphism of the quasigroup $(Z, \star)$. If $\psi = -1$, then we have $-2a = -3b$, $b = (2/3)a$.

Therefore if $a = 3k$, then $Aut(Z, \star) \cong Z_2$. If $a = 3k + 1$ or $a = 3k + 2$, then $Aut(Z, \star)$ consists only of the identity mapping. Thus

$$|Aut(Z, \circ)| = 1.$$

□

Note that all other binary medial quasigroups over the group $(Z, +)$ have one of the following forms: $x * y = x + y + a$, $x \diamond y = x - y + a$, $x \circledast y = -x + y + a$. Every of these quasigroups has an idempotent element: $(-a) * (-a) = -a$, $a \diamond a = a$, $a \circledast a = a$.

Using Lemma 6 and Proposition 4 we can obtain that the automorphism group of any from these quasigroups is isomorphic to the group Z_2. See also [22].

Theorem 5 is similar to Proposition 8 ([18]) that was proved for finite binary T-quasigroups.

Let (Q, f) be an n-T-quasigroup of the form $f(x_1^n) = \sum_{i=1}^n (\varphi_i x_i) + a$ over an abelian group $(Q, +)$. Let $P = \{C(a) - a\} \cap \delta Q$, $C(a)$ be an orbit of the element a in the set Q under action of the group C and C be a centralizer of the elements $\varphi_1, \dots, \varphi_n$ in the group $Aut(Q, +)$, N be a kernel of the endomorphism δ, $\delta = \sum_{i=1}^n \varphi_i - \varepsilon$, S be a stabilizer of the element a under action of the group C on the set Q.

Theorem 5. *If (Q, f) is an n-T-quasigroup of the form $f(x_1^n) = \sum_{i=1}^n (\varphi_i x_i) + a$ over an abelian group $(Q, +)$, and the sets P, N, S have finite order, then*

$$|Aut(Q, f)| = |P| \cdot |N| \cdot |S|.$$

Proof. This proof is a re-written form of the proof of Proposition 8 from [18]. We must do analysis of the equality $\alpha a - a = \delta b$.

If there exists a pair of elements (b, α), $b \in Q, \alpha \in C$ such that the equality $\alpha a - a = \delta b$ is fulfilled, then $L_b^+ \alpha \in Aut(Q, f)$.

Every element $p \in P$ corresponds to $|N|$ elements b_p of the set Q such that $\delta(b_p) = p$ and this element p corresponds to $|S|$ elements ξ_p of the set C such that $\xi_p(a) - a = p$. If $p, r \in P$, $p \neq r$, then $\delta(b_p) \neq \delta(b_r)$ and $\xi_p \neq \xi_r$. Therefore $|Aut(Q, f)| = |P| \cdot |N| \cdot |S|$. □

Corollary 8. *A finite n-ary T-quasigroup (Q, f) of the form $f(x_1^n) = \sum_{i=1}^n (\varphi_i x_i) + a$ over an abelian group $(Q, +)$ has the identity automorphism group if and only if*

$$(Q, +) \cong \bigoplus_{i=1}^{m} (Z_2)_i,$$

$(Q, f) \cong (Q, g)$, where $g(x_1^n) = \sum_{i=1}^n (\varphi_i x_i)$, endomorphism δ is a permutation of the set Q, $|C| = 1$, m is a natural number.

Proof. Let $|Aut(Q, f)| = 1$. From Theorem 5 it follows that in this case $|N| = 1$. Since the order of the set Q is finite, then the endomorphism δ is a permutation of the set Q and it is an automorphism of the group $(Q, +)$.

Further from Lemma 5 it follows that the quasigroup (Q, f) has an idempotent element, $(Q, f) \cong (Q, g)$ where $g(x_1^n) = \sum_{i=1}^n (\varphi_i x_i)$. Moreover, since the endomorphism δ is an automorphism, we obtain that the quasigroup (Q, f) has exactly one idempotent element, and from Corollary 6 it follows that $Aut(Q, f) \cong C$. Therefore in this case we have that $|C| = 1$.

In any abelian group $(Q, +)$ the permutation I, $I(x) = -x$, is an automorphism of this group and $I\psi = \psi I$ for any $\psi \in Aut(Q, +)$, i.e. $I \in C$. Therefore, for a fulfillment of the condition $|C| = 1$ it is necessary that $I = \varepsilon$.

It is well known ([10]) that in finite case only elementary abelian group of order 2^m, where m is a natural number, has the property that $I = \varepsilon$. Therefore $(Q, +) \cong \bigoplus_{i=1}^m (Z_2)_i$.

Conversely. Let (Q, f) be a finite n-ary T-quasigroup of the form $f(x_1^n) \cong \sum_{i=1}^n (\varphi_i x_i)$ over an abelian group $(Q, +) \cong \bigoplus_{i=1}^m (Z_2)_i$, endomorphism δ is a permutation of the set Q and $|C| = 1$.

Then $|N| = 1$, the quasigroup (Q, f) has exactly one idempotent element, by Corollary 6 $Aut(Q, f) \cong C$ and, further, $|Aut(Q, f)| = 1$, since $|C| = 1$. □

Corollary 9. *Any finite medial n-ary quasigroup (Q, f) such that $|Q| \geq 3$ has a non-identity automorphism group.*

Proof. From Theorem 5 it follows that if $|N| > 1$, then $Aut(Q, f) > 1$. If we suppose that $|N| = 1$, then, taking into consideration that quasigroup (Q, f) is finite, we have that endomorphism δ is a permutation of the set Q, quasigroup (Q, f) has exactly one idempotent element and $Aut(Q, f) \cong C$.

Since quasigroup Q is a medial quasigroup, $\varphi_i\varphi_j = \varphi_j\varphi_i$ for all suitable values i, j. Thus $< \varphi_1, \dots, \varphi_n > \subseteq C$.

Then $|C| = 1$ only in case when $\varphi_1 = \dots = \varphi_n = \varepsilon$. In this case quasigroup (Q, f) will have form $f(x_1^n) \cong \sum_{i=1}^n (x_i)$ and by Proposition 2 $Aut(Q, f) \cong Z^{n-1}(Q, +) \rightthreetimes Aut(Q, +)$. It is well known ([10]) that $|Aut(Q, +)| > 1$ for any group $(Q, +)$ such that $|Q| \geq 3$. Therefore $|Aut(Q, f)| > 1$ in this case too. $\square$

Example 3. This example of T-quasigroup with the identity automorphism group is given in [18]. Let $(Q, +) = Z_2 \oplus Z_2 \oplus Z_2$, $x \cdot y = \varphi x + \psi y$, where

$$\varphi = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \psi = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then $|Aut(Q, \cdot)| = 1$.

4. On automorphisms of some isotopes of idempotent n-ary quasigroups

In this section we study some connections between automorphism groups of an idempotent n-quasigroup and some its isotopes.

We denote a set $\{\tau \in Aut(Q, f) | \tau\alpha = \alpha\tau\}$, where (Q, f) is an n-ary quasigroup, α is a permutation of the set Q, by $C_{Aut(Q,f)}(\alpha)$. As it was noted in previous section, the set $C_{Aut(Q,f)}(\alpha)$ forms a group with respect to the usual multiplication of permutations ([10]).

Theorem 6. *If $(Q, f) = (Q, g)T_0$ is an isotope of n-ary idempotent quasigroup (Q, g) such that isotopy T_0 has the form $(\varepsilon, \dots, \varepsilon, \beta_{i+1}, \varepsilon, \dots, \varepsilon)$ (there are $(n + 1)$ members in this sequence) and $i \in \overline{0, n}$, then*

$$Aut(Q, f) = C_{Aut(Q,g)}(\beta_{i+1}).$$

Proof. (a) Let $i \in \overline{0, n-1}$. Let $\varphi \in Aut(Q, f)$, i.e. $\varphi f(x_1, x_2, \dots, x_n) = f(\varphi x_1, \varphi x_2, \dots, \varphi x_n)$ for all $x_1, x_2, \dots, x_n \in Q$. Passing to operation g

we obtain

$$\varphi g(x_1, \dots, x_i, \beta_{i+1}x_{i+1}, x_{i+2}, \dots, x_n) = \\ g(\varphi x_1, \dots, \varphi x_i, \beta_{i+1}\varphi x_{i+1}, \varphi x_{i+2}, \dots, \varphi x_n).$$

If we change x_{i+1} by $\beta_{i+1}^{-1}x_{i+1}$, then further we have

$$\varphi g(x_1, \dots, x_i, x_{i+1}, x_{i+2}, \dots, x_n) = \\ g(\varphi x_1, \dots, \varphi x_i, \beta_{i+1}\varphi\beta_{i+1}^{-1}x_{i+1}, \varphi x_{i+2}, \dots, \varphi x_n). \quad (10)$$

If in (10) $x_1 = x_2 = \dots = x_n = x$, then taking into consideration that operation g is idempotent, we have $\varphi x = g(\varphi x, \dots, \varphi x, \beta_{i+1}\varphi\beta_{i+1}^{-1}x, \varphi x, \dots, \varphi x)$. But $g(\overline{\varphi x}^n) = g(\varphi x, \dots, \varphi x) = \varphi x$. Therefore $g(\varphi x, \dots, \varphi x, \beta_{i+1}\varphi\beta_{i+1}^{-1}x, \varphi x, \dots, \varphi x) = g(\varphi x, \dots, \varphi x) = g(\overline{\varphi x}^n)$. Since (Q, g) is quasigroup from the last equality we have $\beta_{i+1}\varphi\beta_{i+1}^{-1} = \varphi$, then $\beta_{i+1}\varphi = \varphi\beta_{i+1}$.

Therefore we can re-write (10) in the following form $\varphi g(x_1^n) = g(\varphi x_1, \dots, \varphi x_n)$. Then $\varphi \in Aut(Q, g)$, $Aut(Q, f) \subseteq C_{Aut(Q,g)}(\beta_{i+1})$.

Converse. Let $\varphi \in C_{Aut(Q,g)}(\beta_{i+1})$. Then

$$\varphi f(x_1^n) = \varphi g(x_1, \dots, x_i, \beta_{i+1}x_{i+1}, x_{i+2}, \dots, x_n) = \\ g(\varphi x_1, \dots, \varphi x_i, \varphi\beta_{i+1}x_{i+1}, \varphi x_{i+2}, \dots, \varphi x_n) = \\ g(\varphi x_1, \dots, \varphi x_i, \beta_{i+1}\varphi x_{i+1}, \varphi x_{i+2}, \dots, \varphi x_n) = f(\varphi x_1, \varphi x_2, \dots, \varphi x_n).$$

Therefore $C_{Aut(Q,g)}(\beta_{i+1}) \subseteq Aut(Q, f)$ and, finally, we obtain

$$Aut(Q, f) = C_{Aut(Q,g)}(\beta_{i+1}).$$

(b) Let $i = n$. Let $\varphi \in Aut(Q, f)$, i.e. $\varphi f(x_1, x_2, \dots, x_n) = f(\varphi x_1, \varphi x_2, \dots, \varphi x_n)$ for all $x_1, x_2, \dots, x_n \in Q$. Passing to operation g we obtain $\varphi\beta_{n+1}g(x_1, \dots, x_n) = \beta_{n+1}g(\varphi x_1, \dots, \varphi x_n)$.

If in the last equality we put $x_1 = x_2 = \dots = x_n = x$, then, taking into consideration that operation g is idempotent, we have $\varphi\beta_{n+1}x = \beta_{n+1}\varphi x$. Thus

$$\beta_{n+1}\varphi g(x_1, \dots, x_n) = \beta_{n+1}g(\varphi x_1, \dots, \varphi x_n), \\ \varphi g(x_1, \dots, x_n) = g(\varphi x_1, \dots, \varphi x_n), \\ \varphi \in Aut(Q, f), Aut(Q, f) \subseteq C_{Aut(Q,g)}(\beta_{i+1}).$$

Converse. Let $\varphi \in C_{Aut(Q,g)}(\beta_{i+1})$. Then

$$\varphi f(x_1, \dots, x_n) = \varphi\beta_{n+1}g(x_1, \dots, x_n) = \\ \beta_{n+1}\varphi g(x_1, \dots, x_n) = \beta_{n+1}g(\varphi x_1, \dots, \varphi x_n) = \\ = f(\varphi x_1, \varphi x_2, \dots, \varphi x_n).$$

Therefore $C_{Aut(Q,g)}(\beta_{n+1}) \subseteq Aut(Q, f)$.

From inclusions $Aut(Q, f) \subseteq C_{Aut(Q,g)}(\beta_{i+1})$ and $C_{Aut(Q,g)}(\beta_{n+1}) \subseteq Aut(Q, f)$ it follows equality $Aut(Q, f) = C_{Aut(Q,g)}(\beta_{i+1})$. In case (b) this theorem is proved too. □

Corollary 10. *If n-ary quasigroup (Q,f) is an isotope of an n-ary idempotent T-quasigroup $(Q,g), g(x_1^n) = \sum_{i=1}^n \alpha_i x_i$, and the isotopy has the form $(\varepsilon, \dots, \varepsilon,\ \beta_{i+1},\ \varepsilon, \dots, \varepsilon)$, $i \in \overline{0,n}$, $\beta_{i+1} = L_d^+$, then*

$$Aut(Q,f) \cong (Q,+) \rightthreetimes S,$$

where $S = \{\theta \in C | \theta d = d\}$, $C = \{\omega \in Aut(Q,+) \,|\, \omega\alpha_i = \alpha_i\omega\, \forall\, i \in \overline{1,n}\}$.

Proof. From Theorem 6 it follows that we need to find the condition when $L_b\theta L_d = L_d L_b \theta$, where $L_b\theta \in Aut(Q,g)$ (Corollary 7). We have $L_{b+\theta d}\theta = L_{b+d}\theta$, $L_{b+\theta d}\theta 0 = L_{b+d}\theta 0$, $b + \theta d = b + d$, $\theta d = d$. □

Corollary 11. *If $(Q,f) = (Q,g)T_0$ is an isotope of n-ary idempotent T-quasigroup (Q,g) such that isotopy T_0 has the form $(\varepsilon, \dots, \varepsilon,\ \beta_{i+1},$ $\varepsilon, \dots, \varepsilon)$, $i \in \overline{0,n}$ and $\beta_{i+1} = \varphi \in Aut(Q,+)$, then*

$$Aut(Q,f) \cong B \rightthreetimes N$$

where $B = \{L_b^+ | b \in Q, \varphi b = b\}$, $N = \{\sigma \in C | \sigma\varphi = \varphi\sigma\}$.

Proof. From Corollary 7 it follows that any automorphism of n-T-quasigroup (Q,g) has the form $L_b\theta$ where $b \in Q$, L_b is a left translation of abelian group $(Q,+)$, $\theta \in C$. Taking into consideration Theorem 6 we find the condition when the component φ of the isotopy T_0 and an automorphism $L_b\theta$ of quasigroup (Q,g) commute: $\varphi L_b\theta = L_b\theta\varphi$. Further we have $L_{\varphi b}\varphi\theta = L_b\theta\varphi$, $L_{\varphi b}\varphi\theta 0 = L_b\theta\varphi 0$, $\varphi b = b$, $\varphi\theta = \theta\varphi$. □

Remark 7. It is easy to see that in Corollaries 10 and 11 the n-ary quasigroup (Q,f) is an n-ary T-quasigroup.

Corollary 12. *If $x \circ y = \alpha x \cdot y$ where $(Q,\cdot)$ is an idempotent quasigroup, α is a permutation of the set Q, then $Aut(Q,\circ) = C_{Aut(Q,\cdot)}(\alpha)$.*

Proof. If in conditions of Theorem 6 we suppose that $n = 2$, $i = 1$, then we have conditions of this corollary. □

Corollary 13. *If $x \circ y = \gamma(x \cdot y)$ where $(Q,\cdot)$ is an idempotent quasigroup, γ is a permutation of the set Q, then $Aut(Q,\circ) = C_{Aut(Q,\cdot)}(\gamma)$.*

Proof. Proof is analogous to the proof of Corollary 12. □

Example 4. Let $(\mathbb{Q},+)$ be a group of rational numbers. It is known ([10]) that $Aut(\mathbb{Q},+) \cong (\mathbb{Q}^*,\cdot)$, i.e. this group is isomorphic to group of non-zero rational numbers with respect to the operation of multiplication of these numbers. Let $(\mathbb{Q},f)$ be an n-ary medial quasigroup of the form $f(x_1^n) = (\sum_{i=1}^n \varphi_i x_i) + a$, $\varphi_1^n \in \mathbb{Q}^*$, $a \in \mathbb{Q}$.

a). If $\delta = 0$ and $a = 0$, then $Aut(\mathbb{Q}, f) \cong (\mathbb{Q}, +) \leftthreetimes (\mathbb{Q}^*, \cdot)$.
b). If $\delta = 0$ and $a \neq 0$, then $Aut(\mathbb{Q}, f) \cong (\mathbb{Q}, +)$.
c). If $\delta \neq 0$, then $Aut(\mathbb{Q}, f) \cong (\mathbb{Q}^*, \cdot)$.

Proof. a). In this case the quasigroup $(\mathbb{Q}, f)$ is an idempotent quasigroup and by Corollary 7 $Aut(\mathbb{Q}, f) \cong (\mathbb{Q}, +) \leftthreetimes (\mathbb{Q}^*, \cdot)$.

b). By Corollary 10 $Aut(\mathbb{Q}, f) \cong (\mathbb{Q}, +) \leftthreetimes S$. In our case $S = \varepsilon$.

c). In this case the endomorphism δ is a permutation of the set $\mathbb{Q}$, quasigroup $(\mathbb{Q}, f)$ has exactly one idempotent element, by Corollary 6 $Aut(\mathbb{Q}, f) \cong C$. In our case $C = (\mathbb{Q}^*, \cdot)$. Therefore $Aut(\mathbb{Q}, f) \cong (\mathbb{Q}^*, \cdot)$. □

5. On automorphisms of some loop isotopes

In this section we study a connection between automorphism group of a loop and automorphism group of loop isotope of a special form. We denote the identity element of a loop $(Q, +)$ as 0.

Proposition 5. *If $(Q, \circ)$ is a quasigroup with the form $x \circ y = \alpha x + y$, where $(Q, +)$ is a loop, α is a permutation of the set Q such that $\alpha 0 = 0$, then $Aut(Q, \circ) = C_{Aut(Q,+)}(\alpha)$.*

Proof. Let $\varphi \in Aut(Q, \circ)$, i.e. $\varphi(x \circ y) = \varphi x \circ \varphi y$ for all $x, y \in Q$. Passing to operation $+$ we have

$$\varphi(\alpha x + y) = \alpha\varphi x + \varphi y. \tag{11}$$

If we take in equality (11) $x = y = 0$, then $\varphi 0 = \alpha\varphi 0 + \varphi 0$, $\alpha\varphi 0 = 0$, $\varphi 0 = \alpha^{-1} 0 = 0$, therefore $\varphi 0 = 0$. If we assume that $y = 0$ in (11), then $\varphi\alpha x = \alpha\varphi x + \varphi 0 = \alpha\varphi x + 0 = \alpha\varphi x$, i.e. $\varphi\alpha x = \alpha\varphi x$. Therefore $\varphi\alpha = \alpha\varphi$ and $\varphi \in Aut(Q, +)$ since $\varphi(\alpha x + y) = \alpha\varphi x + \varphi y = \varphi\alpha x + \varphi y$. Thus $\varphi \in Aut(Q, +)$, $Aut(Q, \circ) \subseteq C_{Aut(Q,+)}(\alpha)$.

Let $\varphi \in C_{Aut(Q,+)}(\alpha)$. Then $\varphi(x \circ y) = \varphi(\alpha x + y) = \varphi\alpha x + \varphi y = \alpha\varphi x + \varphi y = \varphi x \circ \varphi y$. Therefore $C_{Aut(Q,+)}(\alpha) \subseteq Aut(Q, \circ)$, $C_{Aut(Q,+)}(\alpha) = Aut(Q, \circ)$. □

Proposition 6. *If $(Q, \circ)$ is a quasigroup with the form $x \circ y = x + \beta y$ where $(Q, +)$ is a loop, β is a permutation of the set Q such that $\beta 0 = 0$, then $Aut(Q, \circ) = C_{Aut(Q,+)}(\beta)$.*

Proof. Proof is analogous to the proof of Proposition 5. □

Proposition 7. *If $(Q, \circ)$ is a quasigroup with the form $x \circ y = \gamma(x + y)$, where $(Q, +)$ is a loop, γ is a permutation of the set Q such that $\gamma 0 = 0$,* *then* $\varphi \in Aut(Q, \circ)$ *if and only if: (i)* $(\varphi, \varphi, L_{\varphi 0}\varphi)$ *is an autotopy of the loop* $(Q, +)$*; (ii)* $\gamma^{-1}\varphi\gamma = L_{\varphi 0}\varphi$*; (iii)* $\varphi 0 + x = x + \varphi 0$ *for all* $x \in Q$*; (iv)* $\varphi 0 \circ \varphi 0 = \varphi 0$.

Proof. Let $\varphi \in Aut(Q, \circ)$. Then

$$\varphi\gamma(x + y) = \gamma(\varphi x + \varphi y). \tag{12}$$

If in (12) $x = y = 0$ then $\varphi\gamma 0 = \gamma(\varphi 0 + \varphi 0)$, $\varphi 0 = \gamma(\varphi 0 + \varphi 0)$, $\varphi 0 = \varphi 0 \circ \varphi 0$, i.e. $\varphi 0$ is an idempotent element of the quasigroup $(Q, \circ)$. If we take in (12) $x = 0$, then we have $\varphi\gamma y = \gamma L_{\varphi 0}\varphi y$, $\gamma^{-1}\varphi\gamma = L_{\varphi 0}\varphi$.

By $y = 0$ in (12) we have $\varphi\gamma x = \gamma(\varphi x + \varphi 0)$, $\gamma^{-1}\varphi\gamma = R_{\varphi 0}\varphi$. Then we have

$$\gamma^{-1}\varphi\gamma = R_{\varphi 0}\varphi = L_{\varphi 0}\varphi. \tag{13}$$

We can re-write (12) in the following way $\gamma^{-1}\varphi\gamma(x + y) = \varphi x + \varphi y$ and, taking into consideration equality (13), as $L_{\varphi 0}\varphi(x + y) = \varphi x + \varphi y$, i.e. $(\varphi, \varphi, L_{\varphi 0}\varphi)$ is an autotopy of the loop $(Q, +)$.

From equality (13) it follows that $R_{\varphi 0} = L_{\varphi 0}$, i.e. $x + \varphi 0 = \varphi 0 + x$ for any $x \in Q$.

Conversely. If $(\varphi, \varphi, L_{\varphi 0}\varphi)$ is an autotopy of the loop $(Q, +)$, then we have $L_{\varphi 0}\varphi(x + y) = \varphi x + \varphi y$, $\gamma L_{\varphi 0}\varphi(x + y) = \gamma(\varphi x + \varphi y)$. From (ii) we have $\gamma L_{\varphi 0}\varphi = \varphi\gamma$. $\varphi\gamma(x+y) = \gamma(\varphi x+\varphi y)$, $\varphi(x \circ y) = \varphi x \circ \varphi y$. Therefore $\varphi \in Aut(Q, \circ)$. □

We shall denote by $Z(Q, +)$ the subset of a loop $(Q, +)$ such that

$$Z(Q, +) = \{a \in Q | a + x = x + a \ \forall \ x \in Q\}.$$

Corollary 14. *Let the quasigroup* $(Q, \circ)$ *be an isotope of a loop* $(Q, +)$ *with* $Z(Q, +) = 0$ *of the form* $x \circ y = \gamma(x + y)$ *where* γ *is a permutation of the set* Q *such that* $\gamma 0 = 0$*. Then* $Aut(Q, \circ) = C_{Aut(Q,+)}(\gamma)$.

Proof. If $\varphi \in Aut(Q, \circ)$, then $(\varphi, \varphi, L_{\varphi 0}\varphi)$ is an autotopy of the loop $(Q, +)$. Since $Z(Q, +) = 0$, then $L_{\varphi 0} = \varepsilon$, the permutation φ is an automorphism of the loop $(Q, +)$. From condition (ii) of Proposition 7 it follows that $\gamma\varphi = \varphi\gamma$. Therefore $Aut(Q, \circ) \subseteq C_{Aut(Q,+)}(\gamma)$.

Conversely. Let $\varphi \in C_{Aut(Q,+)}(\gamma)$. Then $\varphi(x \circ y) = \varphi\gamma(x + y) = \gamma\varphi(x + y) = \gamma(\varphi x + \varphi y) = \varphi x \circ \varphi y$. Therefore $Aut(Q, \circ) = C_{Aut(Q,+)}(\gamma)$. □

Remark 8. For groups the condition $Z(Q, +) = 0$ is equivalent to the condition that the centre of the group $(Q, +)$ coincides with 0. In the condition of Corollary 14 the quasigroup $(Q, \circ)$ has exactly one idempotent element.

Proposition 8. *Let the quasigroup* $(Q, \circ)$ *with an unique idempotent element be an isotope of a loop* $(Q, +)$ *of the form* $x \circ y = \alpha x + \beta y$ *where* α, β *are the permutations of the set* Q *such that* $\alpha 0 = \beta 0 = 0$. *Then* $Aut(Q, \circ) = C_{Aut(Q,+)}(\alpha, \beta)$.

Proof. If the quasigroup $(Q, \circ)$ has an unique idempotent element and has form $x \circ y = \alpha x + \beta y$ where $\alpha 0 = \beta 0 = 0$, then $0 \circ 0 = 0$ is this idempotent element. If φ is an automorphism of the quasigroup $(Q, \circ)$, then $\varphi 0 = 0$. Indeed, $\varphi(0 \circ 0) = \varphi 0 \circ \varphi 0$, $\varphi 0 = \varphi 0 \circ \varphi 0$, $\varphi 0 = 0$ because there is only one idempotent element, namely 0.

From $\varphi(x \circ y) = \varphi x \circ \varphi y$ we have $\varphi(\alpha x + \beta y) = \alpha \varphi x + \beta \varphi y$. If we take in the last equality $x = 0$, then $\varphi \beta y = \beta \varphi y$. By analogy $\varphi \alpha = \alpha \varphi$. Then $Aut(Q, \circ) \subseteq C_{Aut(Q,+)}(\alpha, \beta)$.

Further $\varphi(x \circ y) = \varphi(\alpha x + \beta y) = \varphi \alpha x + \varphi \beta y = \alpha \varphi x + \beta \varphi y = \varphi x \circ \varphi y$. Then $Aut(Q, \circ) = C_{Aut(Q,+)}(\alpha, \beta)$. □

Remark 9. It is possible to re-write the condition "the quasigroup $(Q, \circ)$ has an unique idempotent element" as $\alpha x + \beta x \neq x$ for all $x \in Q \setminus \{0\}$.

6. Some known codes as n-ary medial quasigroups and their automorphism groups

In this section we apply obtained results to describe automorphism groups of n-ary quasigroups that correspond to the ISSN code, the EAN code and the UPC code.

Example 5. The International Standard Serial Number code (the ISSN code) which it is used now consists of eight digits. These are the arabic numerals from 0 to 9 on places from the 1-th to the 7-th. On the 8-th place can occur the arabic numerals 0 to 9 and an upper case X. Denote this code as $\mathfrak{C}$.

The first seven digits a_1^7 are the so-called information symbols and the 8-th digit a_8 is a check digit. Any eight right (without any error) digits of the ISSN code satisfy the following check equation:

$$8 \cdot a_1 + 7 \cdot a_2 + 6 \cdot a_3 + 5 \cdot a_4 + 4 \cdot a_5 + 3 \cdot a_6 + 2 \cdot a_7 + 1 \cdot a_8 \equiv 0 \pmod{11},$$

i.e. $a_1^8 \in \mathfrak{C}$ if and only if this code word satisfies the above-stated check equation.

We can associate with the ISSN code 7-ary medial quasigroup (Z_{11}, f) in such manner: $8 \cdot y_1 + 7 \cdot y_2 + 6 \cdot y_3 + 5 \cdot y_4 + 4 \cdot y_5 + 3 \cdot y_6 + 2 \cdot y_7 \equiv -1 \cdot y_8 \pmod{11}$ for all $y_1^7 \in Z_{11}$, $y_8 \equiv 3 \cdot y_1 + 4 \cdot y_2 + 5 \cdot y_3 + 6 \cdot y_4 + 7 \cdot y_5 + 8 \cdot y_6 + 9 \cdot y_7 \pmod{11}$.

Therefore we have 7-ary quasigroup (Z_{11}, f) of the form $f(y_1^7) = 3 \cdot y_1 + 4 \cdot y_2 + 5 \cdot y_3 + 6 \cdot y_4 + 7 \cdot y_5 + 8 \cdot y_6 + 9 \cdot y_7$ over the group $(Z_{11}, +)$. It is easy to see that $f(y_1^7) = y_8$ if and only if $y_1^8 \in \mathfrak{C}$.

Prove that $Aut(Z_{11}, f) \cong Z_{10}$. It is easy to check that the quasigroup (Z_{11}, f) has exactly one idempotent element, namely the element 0. In conditions of this example we can apply Corollary 6. Since $Aut(Z_{11}) \cong Z_{10}$ is a commutative group, we obtain $C \cong Z_{10}$. Therefore

$$Aut(Z_{11}, f) \cong Z_{10}.$$

Example 6. The European Article Number code (the EAN code) is the code with the check equation $1 \cdot x_1 + 3 \cdot x_2 + 1 \cdot x_3 + 3 \cdot x_4 + 1 \cdot x_5 + 3 \cdot x_6 + 1 \cdot x_7 + 3 \cdot x_8 + 1 \cdot x_9 + 3 \cdot x_{10} + 1 \cdot x_{11} + 3 \cdot x_{12} + 1 \cdot x_{13} \equiv 0 \pmod{10}$, where $x_i \in Z_{10}$, $i \in \overline{1, 13}$, elements x_1^{12} are the information digits and element x_{13} is a check digit ([20]).

Similarly as in Example 5 we can associate with this code a 12-ary medial quasigroup (Z_{10}, f). From the last check equation we have $-x_{13} \equiv 1 \cdot x_1 + 3 \cdot x_2 + 1 \cdot x_3 + 3 \cdot x_4 + 1 \cdot x_5 + 3 \cdot x_6 + 1 \cdot x_7 + 3 \cdot x_8 + 1 \cdot x_9 + 3 \cdot x_{10} + 1 \cdot x_{11} + 3 \cdot x_{12} \pmod{10}$, $x_{13} \equiv 9 \cdot x_1 + 7 \cdot x_2 + 9 \cdot x_3 + 7 \cdot x_4 + 9 \cdot x_5 + 7 \cdot x_6 + 9 \cdot x_7 + 7 \cdot x_8 + 9 \cdot x_9 + 7 \cdot x_{10} + 9 \cdot x_{11} + 7 \cdot x_{12} \pmod{10}$.

Therefore we obtain 12-ary medial quasigroup (Z_{10}, f) of the form $f(x_1^{12}) = 9 \cdot x_1 + 7 \cdot x_2 + 9 \cdot x_3 + 7 \cdot x_4 + 9 \cdot x_5 + 7 \cdot x_6 + 9 \cdot x_7 + 7 \cdot x_8 + 9 \cdot x_9 + 7 \cdot x_{10} + 9 \cdot x_{11} + 7 \cdot x_{12}$.

By Proposition 4 $Aut(Z_{11}, f) \cong K \rtimes C$. In conditions of this example we have $K = \{L_0, L_2, L_4, L_6, L_8\}$, $K \cong Z_5$. Since $Aut(Z_{10}) \cong Z_4$ and Z_4 is a commutative group, we obtain $C \cong Z_4$. Therefore

$$Aut(Z_{10}, f) \cong Z_5 \rtimes Z_4.$$

Example 7. The Universal Product Code (the UPC code) is the code with the check equation $1 \cdot 0 + 3 \cdot x_2 + 1 \cdot x_3 + 3 \cdot x_4 + 1 \cdot x_5 + 3 \cdot x_6 + 1 \cdot x_7 + 3 \cdot x_8 + 1 \cdot x_9 + 3 \cdot x_{10} + 1 \cdot x_{11} + 3 \cdot x_{12} + 1 \cdot x_{13} \equiv 0 \pmod{10}$, where $x_i \in Z_{10}$, $i \in \overline{1, 13}$, elements x_1^{12} are the information digits and element x_{13} is a check digit. In other words the UPC code is in fact a subset of the more general the EAN code.

We can associate with this code an 11-ary medial quasigroup (Z_{10}, f) of the form $f(x_2^{12}) = 7 \cdot x_2 + 9 \cdot x_3 + 7 \cdot x_4 + 9 \cdot x_5 + 7 \cdot x_6 + 9 \cdot x_7 + 7 \cdot x_8 + 9 \cdot x_9 + 7 \cdot x_{10} + 9 \cdot x_{11} + 7 \cdot x_{12}$.

We can use Proposition 4 also in this example. We have $K = \{L_0, L_5\}$, $K \cong Z_2$. Since $Aut(Z_{10}) \cong Z_4$, we obtain $C \cong Z_4$. Therefore $Aut(Z_{10}, f) \cong Z_2 \rtimes Z_4$.

Since any element β of the group Z_4 acts on the group Z_2, $Z_2 = \{0, 1\}$, as an inner automorphism ([10]), then $\beta 0 = 0$ and, therefore $\beta 1 = 1$.

Then any element of the group Z_4 acts on the group Z_2 as the identity automorphism, and, finally, we have

$$Aut(Z_{10}, f) \cong Z_2 \times Z_4.$$

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