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On large indecomposable modules, endo-wild representation type and right pure semisimple rings

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Abstract. The existence of large indecomposable right R-modules over a right artinian ring R is discussed in connection with the pure semisimplicity problem and the endo-wildness of the category Mod(R) of right R-modules. Some conjectures and open problems are presented.

1. Introduction

Throughout we assume that R is an associative ring with an identity element. We denote by $J(R)$ the Jacobson radical of R, by Mod(R) the category of all right R-modules and by mod (R) the full subcategory of Mod(R) formed by finitely generated R-modules. Throughout, K denotes a field.

Recall that a ring R is said to be of **finite representation type** if R is both left and right artinian and the number of the isomorphism classes of finitely generated indecomposable right (and left) R-modules is finite. A ring R is said to be **right pure semisimple** if any of the following equivalent conditions is satisfied (see [2], [3], [18], [32], [44], [46], [49], [63], [64])

(P1) *Every right R-module is a direct sum of finitely generated modules.*

(P2) *The right pure global dimension of R is zero.*

(P3) *Every right R-module is algebraically compact.*

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(P4) *The ring R is right artinian and for any sequence*

$$X_1 \xrightarrow{f_1} X_2 \to \cdots \to X_m \xrightarrow{f_m} X_{m+1} \to \cdots$$

of indecomposable modules $X_1, X_2, \ldots$ *in* mod (R) *connected by non-isomorphisms* $f_1, f_2, \ldots$ *there exists* $m \geqslant 2$ *such that*

$$f_m f_{m-1} \cdots f_2 f_1 = 0.$$

It is shown by Auslander [3] that if R is a finite dimensional K-algebra then R is right pure semisimple if and only if R is of finite representation type. Moreover, it was shown in [3] that if R is not right pure semisimple then there exits an indecomposable module in $\mathrm{Mod}(R)$ of infinite length.

One of the aims of this note is to discuss a connection between the right pure semisimplicity of a right artinian ring R and the existence of an indecomposable R-module $X_\aleph$ of cadinality $\geqslant \aleph$, for any arbitrarily large cardinal number $\aleph$. We conjecture in Section 2 that if R is right artinian and R is not right pure semisimple then there exist arbitrarily large indecomposable R-modules, that is, for each infinite cardinal number λ, there exists an indecomposable module in $\mathrm{Mod}(R)$ of cardinality $\geqslant \lambda$ (see Conjecture 1_∞ and Conjecture 2_∞ presented in Section 2 in relation with the Brauer-Thrall conjectures). It follows from Theorem 2.8 that this is the case for a large class of finite dimensional K-algebras R. In connection to this problems we also discuss the endo-wild representation type [59] and the endomorphism ring realisation problem studied by Corner in [8], [9], [10] (see also [21], [22], [59]). We collect in Section 2 several facts related with the existence of large indecomposable modules and we formulate some open problems on right pure semisimple rings. In Section 3 the existence of large indecomposable prinjective modules is discussed. In Section 4 we briefly outline some difficulties in solving the pure semisimplicity problem in relation to the product conjecture.

2. Large indecomposable modules over non pure semisimple rings

For the reader's convenience, we start this section by recalling various characterisations of right pure semisimple rings.

Theorem 2.1. *Let R be a right artinian ring. The following conditions are equivalent.*

(a) *R is right pure semisimple.*

(b) *Every indecomposable right R-module is of finite length.*

(c) *Every (algebraically compact) right R-module is a direct sum of indecomposable modules.*

(d) *There exists a cardinal number λ such that every (algebraically compact) right R-module is a direct sum of modules generated by a set of cardinality at most λ.*

(e) *There exists a module U in $\mathrm{Mod}(R)$ such that every algebraically compact right R-module is a direct summand of a direct sum of copies of U.*

(f) *Every right R-module is a direct sum of modules that are pure-injective or pure-projective.*

(g) *Every right R-module is a direct sum of modules that are pure-injective or countably generated.*

Proof. The equivalence of (a) and (b) is proved in [3]. The equivalence of (a) and (c) is proved in [62, Corollary 2], the equivalence of (a), (c) and (d) is proved in [23, Proposition 10.7], [47, Theorem 1.9], [48, Theorem 1.3] (see also [24, Theorem 3.4]). Finally, the equivalence of (a) and (e) follows from [47, Theorem 1.9], but the equivalence of (a), (f) and (g) is proved in [24, Theorem 3.2]. □

The following characterisation given by Shelah [41] shows that a ring R is right pure semisimple if and only if the Kaplansky's Test Problems for right R-modules have a positive solution (see also [17, Theorem 6]).

Theorem 2.2. *Let R be an associative ring with an identity element. The following conditions are equivalent.*

(a) *The ring R is not right pure semisimple.*

(b) *There is a pair M, N of non-isomorphic right R-modules such that M is isomorphic to a direct summand of N and N is isomorphic to a direct summand of M.*

(c) *There is a pair M, N of non-isomorphic right R-modules such that the modules $M \oplus M$ and $N \oplus N$ are isomorphic.*

(d) *There exists a right module U in $\mathrm{Mod}(R)$ such that, for some $r \geqslant 2$, $U^m \cong U^n$ if and only if $m \equiv n (\mathrm{mod}\ r)$, where U^j means the direct sum of j copies of U.* □

At the origin of the recent developments of representation theory of finite dimensional algebras are the following two conjectures attributed to Brauer and Thrall.

Conjecture 1. *A finite dimensional K-algebra is either representation-finite, or there exist indecomposable modules with arbitrarily large dimension.*

Conjecture 2. *A finite dimensional algebra over an infinite field K is either representation-finite, or there exists an infinite sequence of numbers $d_i \in \mathbb{N}$ such that, for each i, there exists an infinite number of non-isomorphic indecomposable modules of dimension d_i.*

Both of these statements has now be shown to hold true, whenever the field K is algebraically closed. Moreover, the first one also holds true for artin algebras (see [2]).

In [2], Auslander has proved a kind of the Conjecture 1 for artinian rings, by showing that an artinian ring is either of finite representation type, or there exist finitely generated indecomposable right R-modules with arbitrarily large length.

In [36, p. 272], Ringel has shown that the obvious generalisation of the Conjecture 2 for artinian PI rings does not hold. He constructs in [36, 7.5] a hereditary artinian PI ring R such that, for each positive integer n, the number of indecomposable right R-modules of length n is equal to 1 or 2. The ring R is of the form

$$R_M = \begin{pmatrix} F & {}_F M_G \\ 0 & G \end{pmatrix}, \tag{2.3}$$

where F, G are isomorphic division rings and ${}_F M_G$ is a non-simple F-G-bimodule such that $\dim_F M = 2$ and $\dim M_G = 2$. It is easy to see that the ring R_M constructed in [36, 7.5] is not right pure semisimple.

In [54], the author describes a method of construction of a ring of the form R_M such that R_M is right pure semisimple (not a PI ring), the bimodule ${}_F M_G$ is simple, $\dim_F M = \infty$, $\dim M_G = 1$, R_M has two non-isomorphic simple right modules and, for each positive integer $n \geqslant 2$, the number of indecomposable right R_M-modules of length n is equal to 0 or 1.

Since a finite dimensional algebra R is right pure semisimple if and only if R is of finite representation type, then the Brauer-Thrall conjectures, the results in [6] and the observations above suggest the following two conjectures.

Conjecture 1_∞. *A right artinian (or right noetherian) ring R is either* *right pure semisimple or else, for each infinite cardinal λ, there exists an indecomposable R-module of cardinality $\geqslant \lambda$.*

Conjecture $\mathbf{2}_\infty$. *A right artinian (or right noetherian) ring R is either right pure semisimple or else, there exists an infinite sequence $\lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$ of infinite cardinal numbers such that, for each n, the set of the isomorphism classes of the indecomposable right R-modules of cardinality λ_n has the cardinality strictly greater than λ_n.*

Since every indecomposable pure-projective module over any Artin algebra is of finite length (hence pure-injective), and there is an upper bound of the cardinalities of indecomposable pure-injective R-modules, then Conjecture $\mathbf{1}_\infty$ for Artin algebras is just the Problem 3.2 in [56] and can be restated as follows.

Conjecture 3 . *Let R be an Artin algebra. If every indecomposable right R-module is pure-injective then R is of finite representation type.*

Remark. In the conjectures $\mathbf{1}_\infty$ and $\mathbf{2}_\infty$, the assumption that R is artinian or R is noetherian can not be omitted. Indeed, the conjectures are not valid for non-noetherian right semi-artinian V-rings R, because every indecomposable right module over such a ring R is simple injective, and every indecomposable right R-module is simple injective and therefore. We recall that R is a V-ring, if every non-zero right R-module contains a non-zero injective submodule. It was shown by N.V. Dung and P.F. Smith in [16] that the class of non-noetherian algebras which are semi-artinian V-rings is rather large.

There is a close connection of the above problems to the wild, fully wild and endo-wild representation types and to the endomorphism ring realisation problem studied by Corner in [8], [9], [10] (see also [21], [22], [59]). In order to describe it, we denote by $K\langle t_1, t_2\rangle$ the free associative K-algebra of polynomials in two non-commuting indeterminates t_1, t_2 with coefficients in K, and by $\operatorname{modf} K\langle t_1, t_2\rangle$ the category of finite dimensional right $K\langle t_1, t_2\rangle$-modules.

Following [14], [39] and [51] we introduce the following definition.

Definition 2.4. *Let K be a field, Λ a K-algebra and let $\mathcal{C}$ be an additive exact full K-subcategory of the module category $\operatorname{Mod}(\Lambda)$. The category $\mathcal{C}$ is defined to be K-**Wild** (resp. K-**wild**), if there exists an exact K-linear functor $T : \operatorname{Mod} K\langle t_1, t_2\rangle \longrightarrow \mathcal{C}$ (resp. exact K-linear functor $T : \operatorname{modf} K\langle t_1, t_2\rangle \longrightarrow \mathcal{C}$) which respects the isomorphism classes and preserves indecomposables.*

If, in addition, the functor T is full, we call $\mathcal{C}$ **Fully K-Wild** *and* **fully K-wild***, respectively.*

It follows from Lemma 2.5 below, that $\mathcal{C}$ has fully K-wild representation type if and only if it has strictly K-wild representation type in the sense [11], see also below.

Remark. Let us warn the reader that we distinguish between the K-wildness and the wildness. For this purpose, we recall from [11, 8.2] and [36] that the category mod (R) of finite dimensional right modules over a finite dimensional K-algebra R is defined to be **wild** (resp. **strictly wild**), if there exist a finite field extension K' of K and a faithful exact additive functor $T' : \mathrm{modf}\, K'\langle t_1, t_2\rangle \longrightarrow$ mod (R) which respects isomorphism classes and preserves indecomposables (resp. fully faithful, exact, additive functor $T' : \mathrm{modf}\, K'\langle t_1, t_2\rangle \longrightarrow$ mod (R)). In case K is algebraically closed, K-wildness (resp. fully K-wildness) and wildness (resp. strict wildness) of mod (R) coincide, because of the following useful result.

Lemma 2.5. *Let K be a field, Λ a K-algebra and let $\mathcal{C}$ be an additive exact full K-subcategory of the module category* $\mathrm{Mod}(\Lambda)$*.*

(a) *The category $\mathcal{C}$ is of K-wild representation type if and only if there exists an exact K-linear functor $H : \mathrm{modf}\, K\langle t_1, t_2\rangle \longrightarrow \mathcal{C}$, which is faithful, respects the isomorphism classes and preserves indecomposables.*

(b) *The category $\mathcal{C}$ is of fully K-wild representation type if and only if there exists a fully faithful exact K-linear functor*

$$H : \mathrm{modf}\, K\langle t_1, t_2\rangle \longrightarrow \mathcal{C}.$$

In both cases, H has the form $H = (-) \otimes_{K\langle t_1,t_2\rangle} N$, where ${}_{K\langle t_1,t_2\rangle}N_\Lambda$ is a $K\langle t_1, t_2\rangle$-Λ-bimodule such that the left $K\langle t_1, t_2\rangle$-module ${}_{K\langle t_1,t_2\rangle}N$ is free of finite rank.

Proof. The equivalence in (a) is a simple consequence of the Wildness Correction Lemma 2.6 in [51].

(b) The sufficiency is easy, because a fully faithful exact functor $H : \mathrm{modf}\, K\langle t_1, t_2\rangle \longrightarrow \mathcal{C}$ respects the isomorphism classes and preserves indecomposables.

To prove the necessity, assume that $\mathcal{C}$ is of fully K-wild representation type, that is, there exists a fully faithful exact K-linear functor $T : \mathrm{modf}\, K\langle t_1, t_2\rangle \longrightarrow \mathcal{C}$, which respects the isomorphism classes and preserves indecomposables. By the Wildness Correction Lemma 2.6 in [51], there exists an endofunctor $T' : \operatorname{modf} K\langle t_1, t_2\rangle \longrightarrow \operatorname{modf} K\langle t_1, t_2\rangle$ such that the composite functor $T \circ T' : \operatorname{modf} K\langle t_1, t_2\rangle \longrightarrow \mathcal{C}$ satisfies the required conditions and has the form required in the final statement of the lemma. □

Now, following [59, Section 5], we introduce weak substitutes of K-wildness and fully K-wildness for arbitrary additive K-categories as follows.

Definition 2.6. *Let K be an arbitrary field and let $\mathcal{C}$ be an arbitrary additive K-category.*

(a) *$\mathcal{C}$ is defined to be* **weak Fully K-Wild**, *(resp.* **weak fully K-wild**) *if there exists a fully faithful K-linear functor*

$$T : \operatorname{Mod} K\langle t_1, t_2\rangle \longrightarrow \mathcal{C}$$

(resp. fully faithful K-linear functor $T : \operatorname{modf} K\langle t_1, t_2\rangle \longrightarrow \mathcal{C}$).

(b) *$\mathcal{C}$ is defined to be* **K-Endo-Wild**, *(resp.* **K-endo-wild**), *if for each K-algebra A (resp. for each finite dimensional K-algebra A) there is an object U of $\mathcal{C}$ and a K-algebra isomorphism $A \cong \operatorname{End} U$.*

(c) *$\mathcal{C}$ is defined to be* **Corner type K-Endo-Wild**, *(resp.* **Corner type K-endo-wild**), *if for each K-algebra A (resp. for each finite dimensional K-algebra A) there is an object U of $\mathcal{C}$, a nilpotent ideal $\mathfrak{A}$ of $\operatorname{End} U$ and a K-algebra isomorphism $A \cong \operatorname{End} U/\mathfrak{A}$.*

It is clear that Fully K-Wildness and fully K-wildness implies the K-Wildness and K-wildness, as well as weak Fully K-Wildness and weak fully K-wildness, respectively. The following simple observation is very useful.

Lemma 2.7. *Let K be a field and let $\mathcal{C}$ be an arbitrary additive K-category.*

(a) *If $\mathcal{C}$ is weak Fully K-Wild, then $\mathcal{C}$ is K-Endo-Wild.*

(b) *If $\mathcal{C}$ is weak fully K-wild, then $\mathcal{C}$ is K-endo-wild.*

Proof. We recall that for any finite dimensional K-algebra B there exists a fully faithful exact K-linear functor $\operatorname{Mod} B \longrightarrow \operatorname{Mod} K\langle t_1, t_2\rangle$, which restricts to the functor $\operatorname{mod} B \longrightarrow \operatorname{modf} K\langle t_1, t_2\rangle$, see [5], [50, Section 14.2], [51]. On the other hand, by [22, Theorem 1.2], for each K-algebra A there exists a fully faithful exact K-linear functor $\operatorname{Mod} A \longrightarrow$

$\operatorname{Mod}\Gamma_2(K)$, where $\Gamma_2(K) = \begin{pmatrix} K & K^2 \\ 0 & K \end{pmatrix}$ is the Kronecker K-algebra of K-dimension four. Consequently, if $\mathcal{C}$ is weak Fully K-Wild (resp. $\mathcal{C}$ is weak fully K-wild), then for any K-algebra A (resp. any finite dimensional K-algebra A), there exists a fully faithful exact K-linear functor $H : \operatorname{Mod} A \longrightarrow \mathcal{C}$ (resp. fully faithful K-linear functor $H : \operatorname{Mod} A \longrightarrow \mathcal{C}$). Then the functor H induces a K-algebra isomorphism $A \cong \operatorname{End} U$, where $U = H(A)$, and the lemma follows. □

It is known that for any radical square zero K-algebra A of finite dimension, the following two equivalences hold, see [25].

(a) The category $\operatorname{mod} A$ is fully K-wild if and only if $\operatorname{mod} A$ is K-endo-wild.

(b) The category $\operatorname{mod} A$ is K-wild if and only if $\operatorname{mod} A$ is Corner type K-endo-wild.

In this case, for the category $\operatorname{mod} A$, the following representation types coincide: K-endo-wild, weak fully K-wild and fully K-wild representation type.

One of the main aims of this section is to prove the Conjecture 1_∞ for a large class of finite dimensional algebras. To do that, we recall from [4] and [50] that the Jacobson radical of $\operatorname{mod}(R)$ is the two-sided ideal $\operatorname{rad}_R = \operatorname{rad}(\operatorname{mod}(R))$ of the category $\operatorname{mod}(R)$ generated by all non-isomorphisms between indecomposable R-modules. The **infinite radical** $\operatorname{rad}_R^\infty = \operatorname{rad}^\infty(\operatorname{mod}(R))$ of $\operatorname{mod}(R)$ is defined to be the intersection

$$\operatorname{rad}_R^\infty = \bigcap_{j=1}^{\infty} \operatorname{rad}^j(\operatorname{mod}(R))$$

of all powers $\operatorname{rad}^j(\operatorname{mod}(R))$, $j \geqslant 1$, of the radical rad_R of $\operatorname{mod}(R)$ (see [4, p. 179]).

Following Skowroński [61], we call an artin algebra R **loop-finite** if the abelian subgroup $\operatorname{rad}_R^\infty(X,X)$ of the Jacobson radical $\operatorname{rad}_R(X,X)$ of the endomorphism ring $\operatorname{End}_R(X)$ is zero, for all indecomposable modules X in $\operatorname{mod}(R)$.

The class of loop-finite artin algebras is rather large, because according to [61, Section 6] it contains tame tilted algebras, tubular algebras [38] and multicoil algebras [1].

Assume that R has the form $R \cong KQ/I$, where $Q = (Q_0, Q_1)$ is a finite quiver with the set of vertices $Q_0 = \{1, 2, \ldots, n\}$, the set of arrows Q_1 and $I \subset KQ$ is an admissible relation ideal of the path K-algebra KQ of Q (see [4], [50, Chapter 14]).

The algebra $R = KQ/I$ is defined to be **strongly simply connected** if the quiver Q has no oriented cycles and, for any convex subquiver Q' of Q, the path algebra $R' = KQ'/I'$ defined by Q' and the restriction I' of I to Q' has the first Hochschild cohomology group $H^1(R') = H^1(R', R')$ equal to zero, or equivalently, every such an algebra R' has the separation property (see [60, Theorem 4.1]).

Theorem 2.8. *Let K be an algebraically closed field and let R be a finite dimensional K-algebra of any of the following four types:*

(i) *R is loop-finite.*

(ii) *R is strongly simply connected.*

(iii) *The square of the Jacobson radical $J(R)$ of R is zero.*

(iv) *R is the group K-algebra KG of a finite group G.*

Then the following four conditions are equivalent.

(a) *R is of finite representation type.*

(b) *R is right pure semisimple.*

(c) *There exists a cardinal number λ such that any indecomposable right R-module is of cardinality $\leqslant \lambda$.*

(d) *There exists a cardinal number λ (finite or infinite) such that $\dim_K X \leqslant \lambda$, for any indecomposable right R-module.*

Proof. By [3], the conditions (a) and (b) are equivalent. The equivalence of (c) and (d) is an easy exercise. Since (b) obviously implies (c) then it remains to prove that (d) implies (a).

Suppose, to the contrary, that R is of infinite representation type. We prove that, for each infinite cardinal number λ, there exists an indecomposable R-module of dimension $\geqslant \lambda$. We split the proof into three cases.

Case 1° Assume that R is either loop-finite or strongly simply connected. Since R is of infinite representation type then, according to [59, Theorem 1.2], for any K-algebra A there exists a right R-module X and a K-algebra isomorphism

$$A \cong \mathrm{End}_R\, X.$$

Let λ be an infinite cardinal number and let $\mathcal{T}_\lambda = \{t_j\}_{j \in T_\lambda}$ be a fixed set of cardinality 2^{λ^λ}. Denote by

$$A_\lambda = K[\mathcal{T}_\lambda]$$

the polynomial K-algebra in the variables $t_j \in \mathcal{T}_\lambda$ with coefficients in K. Note that $\dim_K A_\lambda \geqslant 2^{\lambda^\lambda}$. By applying the above statement to the K-algebra $A = A_\lambda$, we find a right R-module X_λ and a K-algebra isomorphism $A_\lambda \cong \operatorname{End}_R X_\lambda$. Since A_λ has no non-trivial idempotents then the R-module X_λ is indecomposable.

Note that $\dim_K A_\lambda \leqslant \dim_K \operatorname{End}_K X_\lambda = (\dim_K X_\lambda)^{\dim_K X_\lambda}$. It follows that $\dim_K X_\lambda > \lambda$, because otherwise $\dim_K X_\lambda \leqslant \lambda$ and we get

$$2^{\lambda^\lambda} \leqslant \dim_K A_\lambda \leqslant (\dim_K X_\lambda)^{\dim_K X_\lambda} \leqslant \lambda^\lambda.$$

This contradicts the well-known inequality $2^\aleph > \aleph$, for any cardinal $\aleph$.

Case 2° Assume that R is an arbitrary finite dimensional K-algebra with the Jacobson radical $J(R)$ such that $J(R)^2 = 0$. We set $J = J(R)$. Following Gabriel [19, 9.1], we associate to R the hereditary finite dimensional K-algebra

$$R_J = \begin{pmatrix} R/J & {}_{(R/J)}J_{(R/J)} \\ 0 & R/J \end{pmatrix}$$

and the reduction functor (see also [4])

$$\mathbb{F} : \operatorname{Mod}(R) \longrightarrow \operatorname{Mod}(R_J)$$

defined by attaching to any module Y in $\operatorname{Mod}(R)$ the triple $\mathbb{F}(Y) = (Y', Y'', t)$, where $Y' = Y/YJ$, $Y'' = YJ$ are viewed as right R/J-modules and $t : Y' \otimes_{R/J} J_{R/J} \to Y''_{R/J}$ is an R/J-module homomorphism defined by the formula $t(\overline{y} \otimes r) = y \cdot r$ for $\overline{y} = y + J$ and $r \in J$. The triple $\mathbb{F}(Y)$ is viewed as a right R_J-module in a natural way. If $f : Y \to Z$ is an R-module homomorphism we set $\mathbb{F}(f) = (f', f'')$, where $f'' : Y'' \to Z''$ is the restriction of f to $Y'' = YJ$ and $f' : Y' \to Z'$ is the R/J-module homomorphism induced by f.

The functor $\mathbb{F}$ has the following properties.

(i) $\mathbb{F}$ is full and establishes a representation equivalence between $\operatorname{Mod}(R)$ and the category $\operatorname{Im}\mathbb{F}$, that is, a homomorphism $f : Y \to Z$ in $\operatorname{Mod}(R)$ is an isomorphism if and only if $\mathbb{F}(f)$ is an isomorphism in $\operatorname{Mod}(R_J)$. Moreover, $\mathbb{F}$ restricts to the functor $\mathbb{F} :\ \operatorname{mod}\,(R) \longrightarrow \operatorname{mod}\,(R_J)$.

(ii) A right R_J-module M belongs to $\operatorname{Im}\mathbb{F}$ if and only if M has no non-zero summand isomorphic to a simple projective right R_J-module.

(iii) The functor $\mathbb{F}$ carries a homomorphism $f : Y \to Z$ in $\mathrm{Mod}(R)$ to zero if and only if $\mathrm{Im} f \subseteq ZJ$. In particular, for any R-module Y, the algebra homomorphism $\mathbb{F}_Y : \mathrm{End}\, Y \longrightarrow \mathrm{End}\, \mathbb{F}(Y)$, given by $f \mapsto \mathbb{F}(f)$, is surjective, $\mathrm{Ker}\mathbb{F}_Y = \mathrm{Hom}_R(Y, YJ)$ and $(\mathrm{Ker}\mathbb{F}_Y)^2 = 0$.

(iv) The functor $\mathbb{F}$ preserves the indecomposability, projectivity and the dimension of modules. Moreover, $\mathbb{F}$ defines a bijection between the isomorphism classes of indecomposable modules in $\mathrm{Mod}(R)$ and the isomorphism classes of indecomposable modules in $\mathrm{Mod}(R_J)$, which are not simple nor projective.

(v) The ring R is right pure semisimple (resp. of finite representation type) if and only if R_J is right pure semisimple (resp. of finite representation type).

The statements (i)–(iv) are essentially proved in [19, Section 9] (see also [4, Lemma X.2.1]).

For the proof of (v), we note that, by (ii)–(iii), the ring R is of finite representation type if and only if R_J of finite representation type. To finish the proof of (v) we recall from [44] and [46] that a right artinian ring S is right pure semisimple if and only if the ideal $\mathrm{rad}_R = \mathrm{rad}(\ \mathrm{mod}\, R)$ is right T-nilpotent, that is, for every sequence

$$X_1 \xrightarrow{f_1} X_2 \to \cdots \to X_m \xrightarrow{f_m} X_{m+1} \to \cdots$$

of indecomposable modules $X_1, X_2, \ldots$ in $\mathrm{mod}\, R$ connected by nonisomorphisms $f_1, f_2, \ldots$ there exists $m \geqslant 2$ such that $f_m f_{m-1} \cdots f_2 f_1 = 0$. Hence, in view of (i)–(iv), the ring R is right pure semisimple if and only if R_J is right pure semisimple. The details are left to the reader.

To finish the proof in the Case 2°, we note that since the algebra $S = R/J$ is semisimple, then the algebra R_J is hereditary. Since R is of infinite representation type then, by (v), the algebra R_J is also of infinite representation type. We claim that there exists a fully faithful exact K-linear functor

$$\mathbb{G} : \mathrm{Mod}\,\Gamma_2(K) \longrightarrow \mathrm{Mod}\, R_J,$$

where $\Gamma_2(K) = \begin{pmatrix} K & K^2 \\ 0 & K \end{pmatrix}$ is the Kronecker K-algebra. This is obvious when R_J is of K-wild representation type (see [50, Chapter 14] for definition), because R_J is hereditary and, according to [11, Theorem 8.4], the category $\mathrm{mod}\, R_J$ is of fully K-wild representation type and [51, Theorem 2.9] applies. If the algebra R_J is of tame representation type (see [50, Chapter 14] for definition), the existence of the functor $\mathbb{G}$ follows from [13].

It follows from [22] and the properties of $\mathbb{G}$ that, for any K-algebra A, there exists a right R_J-module X and a K-algebra isomorphism

$$A \cong \operatorname{End}_{R_J} X,$$

because we know from [22] that any K-algebra A has the form $A \cong \operatorname{End}_{\Gamma_2(K)} U$, where U is a right $\Gamma_2(K)$-module.

Let λ be an infinite cardinal number. By the arguments used in the Case 1°, there exists an indecomposable R_J-module M_λ such that $\dim_K M_\lambda \geqslant \lambda$. Since M_λ is not simple projective then, according to (iv), there exists an indecomposable R-module X_λ such that $\mathbb{F}(X_\lambda) \cong M_\lambda$ and $\dim_K X_\lambda = \dim_K M_\lambda \geqslant \lambda$. This finishes the proof in Case 2°.

Case 3° Assume that R is the group K-algebra KG, where G a finite group. Since R is of infinite representation type, then the characteristic p of K is a prime, p divides the order of G and according to Higman's theorem [26] a p-Sylow subgroup H of G is not cyclic. It follows that there is a group epimorphism $H \to H'$, where H' is the direct sum of two copies of the cyclic group of order p (see [12, 6.10]). Since

$$KH' \cong K[t_1,t_2]/(t_1^p - 1, t_2^p - 1) \cong K[t_1,t_2]/(t_1^p, t_2^p),$$

then there are K-algebra surjections

$$KG \to KH \to KH' \to K[t_1,t_2]/(t_1,t_2)^p \to S,$$

where $S = K[t_1,t_2]/(t_1,t_2)^2$. The composite K-algebra surjection $KG \to S$ induces a fully faithful exact embedding

$$\operatorname{Mod}(S) \longrightarrow \operatorname{Mod}(R).$$

Since $J(S)^2 = 0$ and S is of infinite representation type, then the Case 2° applies to S and we are done. This completes the proof of the theorem. □

Corollary 2.9. *Let K be an algebraically closed field and let R be a finite dimensional K-algebra which is loop-finite or strongly simply connected. Then the following four conditions are equivalent.*

(a) *R is of infinite representation type.*

(b) *For any cardinal number λ there exists an indecomposable right R-module X such that $\dim_K X > \lambda$.*

(c) *The category* Mod R *is Fully* K*-Wild.*

(d) *The category* Mod R *is* K*-Endo-Wild, that is, any* K*-algebra* A *has, up to isomorphism, the form* $A = \operatorname{End}_R X$*, for some right* R*-module* X*.*

Proof. By Theorem 2.8, the conditions (a) and (b) are equivalent.

To prove that (a) implies (c), assume that the algebra R is loop-finite or strongly simply connected. It follows from Corollary 2.2 and Theorem 3.1 in [59] that there exists a fully faithful exact K-linear functor $\operatorname{Mod}\Gamma_2(K) \longrightarrow \operatorname{Mod} R$ which restricts to the functor $\operatorname{mod}\Gamma_2(K) \longrightarrow \operatorname{mod} R$, where $\Gamma_2(K)$ is the Kronecker K-algebra. On the other hand, by [22, Theorem 1.2], for any K-algebra A there exists a fully faithful exact K-linear functor $\operatorname{Mod} A \longrightarrow \operatorname{Mod}\Gamma_2(K)$. It follows that the category Mod R is Fully K-Wild.

Since (c) implies (d), by Corollary 2.7, it remains to prove that (d) implies (a). By (d) applied to the polynomial algebra $A = K[t]$, there exists a right R-module X such that $\operatorname{End} X \cong K[t]$. It follows that R is not right pure semisimple, because the module X is indecomposable of infinite K-dimension. By Theorem 2.8, R is of infinite representation type. This finishes the proof. □

We now collect some questions and open problems related with the results and conjectures stated above.

Questions and problems 2.10. (P1) We do not know if Theorem 2.8 and Corollary 2.9 remain valid in case the field K is not algebraically closed and R is a loop-finite K-algebra. By [59, Theorem 2.1], the proof reduces to the question if the category Mod R_M is Fully K-Wild, where R_M is a finite dimensional hereditary K-algebra of the triangular form $R_M = \begin{pmatrix} F & {}_F M_G \\ 0 & G \end{pmatrix}$, F, G are division K-algebras and ${}_F M_G$ is a simple F-G-bimodule such that $\dim_F M \cdot \dim M_G = 4$.

(P2) Suppose that K is algebraically closed and R is a loop-finite or strongly simply connected K-algebra. It would be interesting to know if some of the following conditions are equivalent:

(a) the category mod R is K-wild,

(b) the category mod R is fully K-wild,

(c) the category mod R is weak fully K-wild,

(d) the category mod R is K-endo-wild,

(e) the category mod R is Corner type K-endo-wild.

(P3) The validity of Conjecture 1° remains an open problem. We are not able to prove it even for hereditary rings of the form R_M shown in (2.3), where ${}_FM_G$ is a simple F-G-bimodule. Note that, in case the bimodule ${}_FM_G$ is not simple, the results of Ringel [36] apply.

(P4) Is a semiperfect ring R right artinian or right pure semisimple if every indecomposable right R-module is pure-injective or pure-projective?

(P5) Let R be a ring and suppose that there exists a cardinal number $\aleph$ such that every (pure-injective) right R-module is a direct sum of modules that are pure-injective or $\aleph$-generated. Is R right pure-semisimple? (We do not know the answer even for $\aleph = \aleph_1$).

(P6) Let R be a right noetherian (or right artinian) ring and suppose that the isomorphism classes of the indecomposable right R-modules form a set. Is R right pure semisimple?

(P7) Let G be a group and Λ a G-graded artinian ring (or a finite dimensional algebra over a field K). Let $\mathrm{Mod}^G(\Lambda)$ be the Grothendieck category of G-graded right Λ-modules. Give necessary and sufficient conditions for G and Λ to be $\mathrm{Mod}^G(\Lambda)$ pure semisimple, compare with [20, Example 4].

3. The existence of large prinjective modules

Now we show an analogue of Theorem 2.8 for the category $\mathrm{Prin}(KI)$ of prinjective right modules over the incidence K-algebra KI of a finite poset $(I, \preceq)$ with coefficients in K (see [50]). We recall from [52] that the algebra KI is of finite global dimension, the right socle of KI is projective and the simple projective right ideals are of the form e_pKI, up to isomorphism, where p runs through the set $\max I$ of all maximal elements of the poset I. We recall from [52] that a right KI-module X is said to be **prinjective** if the projective dimension of X is at most 1 and there exists a short exact sequence

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

in $\mathrm{Mod}(KI)$ such that the module P_0 is projective and P_1 is semisimple projective. We denote by $\mathrm{Prin}(KI)$ the full subcategory of $\mathrm{Mod}(KI)$ consisting of all prinjective modules, and by $\mathrm{prin}(KI)$ the full subcategory of $\mathrm{Prin}(KI)$ consisting of finite dimensional modules. Together with the hereditary subcategory $\mathrm{Prin}(KI)$ of $\mathrm{Mod}(KI)$ we also study the full subcategory $\mathrm{Mod}_{sp}(KI)$ of $\mathrm{Mod}(KI)$ formed by all modules X such that the socle $\mathrm{soc}X$ of X is a projective module. Denote by $\mathrm{mod}_{sp}(KI)$ the full subcategory of $\mathrm{Mod}_{sp}(KI)$ consisting of finite dimensional modules. We recall from [52] that there is a full and dense K-linear functor

$$\Theta : \mathrm{prin}(KI) \longrightarrow \mathrm{mod}_{sp}(KI)$$

preserving finite, tame and K-wild representation type. The functor Θ vanishes only on finitely many indecomposable KI-modules, namely the projective KI^--modules $e_j KI^-$, where $j \in I^- = I \setminus \max I$ and $e_j \in KI$ is the standard matrix idempotent corresponding to $j \in I^-$. The kernel of Θ consists of all the KI-homomorphisms $f : X \to Y$ that have a factorisation through a direct sum of copies of the modules $e_j KI^-$, with $j \in I^-$.

The additive category $\mathrm{mod}_{sp}(KI)$ is not abelian, has enough projectives, the global homological dimension of $\mathrm{mod}_{sp}(KI)$ is finite and equals gl.dim KI. The categories $\mathrm{prin}(KI)$ and $\mathrm{mod}_{sp}(KI)$ are playing an important role in representation theory of algebras and matrix problems (see [50] and [52]).

The idea of the proof of Theorem 2.8 leads to the following results.

Theorem 3.1. *Let K be an arbitrary field, let I be a finite poset and let KI be the incidence K-algebra of I with coefficients in K. The following conditions are equivalent.*

(a) *The category* $\mathrm{prin}(KI)$ *has only a finite number of indecomposable modules, up to isomorphism.*

(b) *$q_I(v) > 0$, for any non-zero vector $v \in \mathbb{N}^I$, where $q_I : \mathbb{Z}^I \longrightarrow \mathbb{Z}$ is the Tits quadratic form*

$$q_I(x) = \sum_{i\in I} x_i^2 + \sum_{\substack{i \prec j \\ j \notin \max I}} x_i x_j - \sum_{p \in \max I} \Big(\sum_{i \prec p} x_i\Big) x_p$$

of the poset I, see [52].

(c) *There exists a cardinal number λ such that any indecomposable KI-module in* $\mathrm{Prin}(KI)$*is of cardinality $\leqslant \lambda$.*

(d) *There exists a cardinal number λ (finite or infinite) such that $\dim_K X \leqslant \lambda$, for any indecomposable module X in* $\mathrm{Prin}(KI)$.

(d′) *There exists a cardinal number λ (finite or infinite) such that $\dim_K X \leqslant \lambda$, for any indecomposable module X in* $\mathrm{Mod}_{sp}(KI)$.

Proof. The equivalence of (a) and (b) is proved in [52], and equivalence of (c) and (d) is an easy exercise. Since (a) obviously implies (d), and the equivalence of (d) and (d′) easily follows from the properties of the functor $\Theta : \mathrm{Prin}(KI) \longrightarrow \mathrm{Mod}_{sp}(KI)$ proved in [52], it remains to prove that (d) implies (a).

Suppose, to the contrary, that the category $\mathrm{prin}(KI)$ has an infinite number of indecomposable modules, up to isomorphism. It follows from [52] that the Tits quadratic form $q_I : \mathbb{Z}^I \longrightarrow \mathbb{Z}$ is not weakly positive.

We prove that, for each infinite cardinal number λ, there exists an indecomposable module in $\mathrm{Prin}(KI)$ of dimension $\geqslant \lambda$.

Since q_I is not weakly positive then, according to [21, Theorem 1.8], any K-algebra A is isomorphic to an algebra of the form $\mathrm{End}\, X$, where X is a module in $\mathrm{Prin}(KI)$.

Let λ be an infinite cardinal number and let $\mathcal{T}_\lambda = \{t_j\}_{j \in T_\lambda}$ be a fixed set of cardinality 2^{λ^λ}. Then, for the polynomial K-algebra $A_\lambda = K[\mathcal{T}_\lambda]$ in the variables $t_j \in \mathcal{T}_\lambda$ with coefficients in K, there exists a module X_λ in $\mathrm{Prin}(KI)$ and a K-algebra isomorphism $A_\lambda \cong \mathrm{End}_{KI} X_\lambda$. Since A_λ has no non-trivial idempotents then the R-module X_λ is indecomposable. One shows, as in the proof of Theorem 2.8, that $\dim_K X_\lambda > \lambda$. This finishes the proof of the theorem. □

The following corollary is a consequence of Theorem 3.1 and the arguments used in the proof of Corollary 2.9.

Corollary 3.2. *Let K be an arbitrary field, I a finite poset and let KI be the incidence K-algebra of I with coefficients in K. Let $q_I : \mathbb{Z}^I \longrightarrow \mathbb{Z}$ be the Tits quadratic form of I. The following conditions are equivalent.*

- (a) *The category* $\mathrm{prin}(KI)$ *has an infinite number of indecomposable modules, up to isomorphism.*
- (b) *There exists a non-zero vector $v \in \mathbb{N}^I$ such that $q_I(v) = 0$.*
- (c) *For any cardinal number λ there exists an indecomposable X in* $\mathrm{Prin}(KI)$ *such that* $\dim_K X > \lambda$.
- (c′) *For any cardinal number λ there exists an indecomposable X in* $\mathrm{Mod}_{sp}(KI)$ *such that* $\dim_K X > \lambda$.
- (d) *The category* $\mathrm{Prin}(KI)$ *is Fully K-Wild.*
- (d′) *The category* $\mathrm{Mod}_{sp}(KI)$ *is weak Fully K-Wild.*
- (e) *The category* $\mathrm{Prin}(KI)$ *is K-Endo-Wild.*

(e′) *The category* $\mathrm{Mod}_{sp}(KI)$ *is* K*-Endo-Wild.*

Proof. By Theorem 3.1, the conditions (a), (c) and (c′) are equivalent. Moreover, the conditions (a) and (b) are equivalent, because, by [52, Theorem 3.1], the existence of a non-zero vector $v \in \mathbb{N}^I$ such that $q_I(v) \leqslant 0$ implies that I contains, as a peak subposet, any of the critical posets $\mathcal{P}_1, \dots, \mathcal{P}_{114}$ listed in [52, Section 5], say I contains $\mathcal{P}_j$. Let $\boldsymbol{\mu}_{\mathcal{P}_j} \in \mathbb{N}^{\mathcal{P}_j}$ be the vector in $\mathrm{Ker} q_{\mathcal{P}_j}$, shown in [52, Section 5], such that $\mathrm{Ker} q_{\mathcal{P}_j} = \mathbb{Z}\boldsymbol{\mu}_{\mathcal{P}_j}$. Then $q_I(\overline{\boldsymbol{\mu}}_{\mathcal{P}_j}) = 0$, where $\overline{\boldsymbol{\mu}}_{\mathcal{P}_j} \in \mathbb{N}^I$ is an obvious extension of $\boldsymbol{\mu}_{\mathcal{P}_j}$ defined by the formulae $(\overline{\boldsymbol{\mu}}_{\mathcal{P}_j})_s = (\boldsymbol{\mu}_{\mathcal{P}_j})_s$ for $s \in \mathcal{P}_j$, and $(\overline{\boldsymbol{\mu}}_{\mathcal{P}_j})_s = 0$ for $s \notin \mathcal{P}_j$.

Now we prove the implications (a)⇒(d) and (a)⇒(d′). Assume that the category $\mathrm{prin}(KI)$ is of infinite representation type. It follows from [21, Theorem 1.7] and [31, Theorem 3.16] that there exist a fully faithful K-linear functors

$$H : \mathrm{Mod}\,\Gamma_2(K) \longrightarrow \mathrm{Prin}(KI) \quad \text{and} \quad H' : \mathrm{Mod}\,\Gamma_2(K) \longrightarrow \mathrm{Mod}_{sp}(KI),$$

where $\Gamma_2(K)$ is the Kronecker K-algebra, and the functor H is exact. On the other hand, by [22, Theorem 1.2], for any K-algebra A there exists a fully faithful exact K-linear functor $\mathrm{Mod}\,A \longrightarrow \mathrm{Mod}\,\Gamma_2(K)$. It follows that the category $\mathrm{Prin}(KI)$ is Fully K-Wild and the category $\mathrm{Mod}_{sp}(KI)$ is weak Fully K-Wild.

The implications (d)⇒(e) and (d′)⇒(e′) are a consequence of Corollary 2.7. Since the remaining implications (e)⇒(a) and (e′)⇒(a) follow in a similar way as (d)⇒(a) in the proof of Corollary 2.9, the proof is complete. □

We finish this section with a discussion of K-endo-wildness of the category $\mathrm{mod}_{sp}(KI)$.

Proposition 3.3. *Let* K *be an arbitrary field,* I *a finite poset and let* KI *be the incidence* K*-algebra of* I *with coefficients in* K*. Let* $q_I : \mathbb{Z}^I \longrightarrow \mathbb{Z}$ *be the Tits quadratic form of* I*. The following conditions are equivalent.*

(a) *The category* $\mathrm{prin}(KI)$ *is fully* K*-wild if and only if there exists a non-zero vector* $v \in \mathbb{N}^I$ *such that* $q_I(v) = -1$.

(b) *If the category* $\mathrm{prin}(KI)$ *is fully* K*-wild then the category* $\mathrm{mod}_{sp}(KI)$ *is weak fully* K*-wild and* K*-endo-wild.*

Outline of proof. (a) Assume that the category $\mathrm{prin}(KI)$ is fully K-wild. By [30, Theorem 1.3], the poset I contains, as a peak subposet, a poset $\mathcal{H}$ isomorphic to one of the hypercritical posets listed in [29, Table 2], up to peak-reduction procedure. It is shown in [29, Lemma 5.6] that each such a poset has the form $\mathcal{H} = \mathcal{P} \cup \{a\}$, where $\mathcal{P}$ is isomorphic to one of the critical posets $\mathcal{P}_1, \ldots, \mathcal{P}_{110}$ shown in [29, Table 1] and $a \in \mathcal{H}$ is a point (marked as a black point in [29, Table 2]) such that the index

$$\mathrm{ind}_{\mathcal{P}}^{\mathcal{H}}(a) = 2(\eta_a, \overline{\boldsymbol{\mu}}_{\mathcal{P}})_{\mathcal{H}}$$

is negative, where $\eta_a \in \mathbb{Z}^{\mathcal{H}}$ is the a-th standard basis vector, $\boldsymbol{\mu}_{\mathcal{P}} \in \mathbb{Z}^{\mathcal{P}}$ is the vector shown in [29, Table 1] such that $\mathrm{Ker}\, q_{\mathcal{P}} = \mathbb{Z}\boldsymbol{\mu}_{\mathcal{P}}$, $(-,-)_{\mathcal{H}} : \mathbb{Z}^{\mathcal{H}} \times \mathbb{Z}^{\mathcal{H}} \longrightarrow \mathbb{Z}$ is the symmetric $\mathbb{Z}$-bilinear form associated to the Tits form $q_{\mathcal{H}}$ and $\overline{\boldsymbol{\mu}}_{\mathcal{H}} \in \mathbb{Z}^{\mathcal{H}}$ is defined by the formulae $\overline{\boldsymbol{\mu}}_{\mathcal{H}}(t) = \boldsymbol{\mu}_{\mathcal{P}}(t)$ for $t \in \mathcal{P}$, and $\overline{\boldsymbol{\mu}}_{\mathcal{H}}(a) = 0$.

A case by case inspection of the posets in [29, Table 2] shows that $\mathrm{ind}_{\mathcal{P}}^{\mathcal{H}}(a) = -2$, if $\mathcal{H}$ is one of the posets $\mathcal{N}_1^*$, $\widetilde{\widetilde{\mathbb{A}}}{}^*_{3,1}$, $\widetilde{\widetilde{\mathbb{A}}}{}^*_{3,2}$, $\widetilde{\widetilde{\mathbb{D}}}{}^*_{4,2}$, $\widetilde{\widetilde{\mathbb{D}}}{}^*_{n+6,8}$ and $\mathcal{H}_{24}$, whereas $\mathrm{ind}_{\mathcal{P}}^{\mathcal{H}}(a) = -1$ for the remaining posets listed in [29, Table 2]. Define the vector $\mathbf{w}_{\mathcal{H}} \in \mathbb{Z}^{\mathcal{H}}$ by the formula

$$\mathbf{w}_{\mathcal{H}} = \begin{cases} \eta_a + \overline{\boldsymbol{\mu}}_{\mathcal{H}}; & \text{if } \mathrm{ind}_{\mathcal{P}}^{\mathcal{H}}(a) = -2, \\ \eta_a + 2\overline{\boldsymbol{\mu}}_{\mathcal{H}}; & \text{if } \mathrm{ind}_{\mathcal{P}}^{\mathcal{H}}(a) = -1. \end{cases}$$

It follows that $q_{\mathcal{H}}(\mathbf{w}_{\mathcal{H}}) = -1$. Indeed, if $\mathrm{ind}_{\mathcal{P}}^{\mathcal{H}}(a) = -2$, then $\mathbf{w}_{\mathcal{H}} = \eta_a + \overline{\boldsymbol{\mu}}_{\mathcal{H}}$ and

$$q_{\mathcal{H}}(\mathbf{w}_{\mathcal{H}}) = (\mathbf{w}_{\mathcal{H}}, \mathbf{w}_{\mathcal{H}})_{\mathcal{H}} = q_{\mathcal{H}}(\overline{\boldsymbol{\mu}}_{\mathcal{H}}) + \mathrm{ind}_{\mathcal{P}}^{\mathcal{H}}(a) + q_{\mathcal{H}}(\eta_a) = -2 + 1 = -1.$$

Similarly, if $\mathrm{ind}_{\mathcal{P}}^{\mathcal{H}}(a) = -1$, then $\mathbf{w}_{\mathcal{H}} = \eta_a + 2\overline{\boldsymbol{\mu}}_{\mathcal{H}}$ and

$$q_{\mathcal{H}}(\mathbf{w}_{\mathcal{H}}) = (\mathbf{w}_{\mathcal{H}}, \mathbf{w}_{\mathcal{H}})_{\mathcal{H}} = 4q_{\mathcal{H}}(\overline{\boldsymbol{\mu}}_{\mathcal{H}}) + 2\mathrm{ind}_{\mathcal{P}}^{\mathcal{H}}(a) + q_{\mathcal{H}}(\eta_a) = -2 + 1 = -1.$$

Since the poset I contains $\mathcal{H}$, as a peak subposet, then obviously $q_I(\overline{\mathbf{w}}_{\mathcal{H}}) = -1$, where $\overline{\mathbf{w}}_{\mathcal{H}} \in \mathbb{N}^I$ is an obvious extension of the corresponding vector $\mathbf{w}_{\mathcal{H}}$. The converse implication in the statement (a) follows from [30, Theorem 1.3].

(b) Assume that the category $\mathrm{prin}(KI)$ is fully K-wild and let $\mathcal{H}$ be a hypercritical peak subposets of I chosen as in the proof of (a) above. A case by case inspection of the poset listed in [30, Table 1] shows that there exists a minimal element $a \in \mathcal{H}^- = \mathcal{H} \setminus \max\mathcal{H}$ such that

(1) the subposet $a^\triangle = \{j \in \mathcal{H}^-; j \succeq a\}$ of $\mathcal{H}^-$ is linearly ordered,

(2) the elements of $a^\triangle$ are incomparable with all elements of $\mathcal{H}^- \setminus a^\triangle$, and

(3) the subposet $\mathcal{H}'_a = \mathcal{H} \setminus \{a\}$ of $\mathcal{H}$ is of infinite prinjective type and every proper subposet of $\mathcal{H}'_a$ is of finite prinjective type.

Consider the $\mathbb{Z}$-linear map $\widehat{\ell}_a : \mathbb{Z}^{\mathcal{H}} \longrightarrow \mathbb{Z}$ defined by the formula

$$\widehat{\ell}_a(v) = \sum_{a \prec p \in \max \mathcal{H}} \sum_{i \prec p} v(i) - \sum_{i \succ a} v(i), \tag{3.4}$$

for any $v \in \mathbb{Z}^I$, see [31, (3.5)].

Following the idea of the proof of Lemma 3.8 in [30] and of Theorem 3.6 in [31] (or by applying Corollary 3.17 in [28]) one can show that there exists an indecomposable non-projective module U in $\mathrm{prin}(K\mathcal{H}'_a) \cap \mathrm{mod}_{sp} K\mathcal{H}'_a$ such that $\mathrm{End}_{K\mathcal{H}'_a} U \cong K$ and $\widehat{\ell}_a(\mathbf{cdn}\widehat{U}) \geqslant 3$, where $\widehat{U}$ is a $K\mathcal{H}$-module obtained from U by completing it with the zero space over a and $\mathbf{cdn}\widehat{U} \in \mathbb{N}^I$ is the coordinate vector of the module $\widehat{U}$, see [52]. It follows from [31, Lemma 3.4] that

- the indecomposable $K\mathcal{H}$-module $P_a = e_a K\mathcal{H}$ is hereditary projective and belongs to $\mathrm{prin}(K\mathcal{H}) \cap \mathrm{mod}_{sp} K\mathcal{H}'_a$,
- the module $\widehat{U}$ belongs to $\mathrm{prin}(K\mathcal{H}) \cap \mathrm{mod}_{sp} K\mathcal{H}'_a$,
- $\mathrm{Hom}_{K\mathcal{H}}(\widehat{U}, P_a) = 0$, $\mathrm{Hom}_{K\mathcal{H}}(P_a, \widehat{U}) = 0$, $\mathrm{End}\, P_a \cong K$ and $\mathrm{End}\, \widehat{U} \cong \mathrm{End}\, U \cong K$, and
- $\dim_K \mathrm{Ext}^1_{K\mathcal{H}}(\widehat{U}, P_a) = \widehat{\ell}_a(\mathbf{cdn}\widehat{U}) \geqslant 3$.

It follows that for $V = P_a$ and $\widehat{U}$ we get a fully faithful exact K-linear functor $T_{V,\widehat{U}} : \mathrm{mod} \begin{pmatrix} K & K^3 \\ 0 & K \end{pmatrix} \longrightarrow \mathrm{mod}(K\mathcal{H})$. Since the modules $V = P_a$ and $\widehat{U}$ are both prinjective and socle projective and the subcategory $(\mathrm{prin} K\mathcal{H}) \cap \mathrm{mod}_{sp} K\mathcal{H}$ of $\mathrm{mod}(K\mathcal{H})$ is closed under forming extensions, then the image of $T_{V,\widehat{U}}$ is contained in $(\mathrm{prin} K\mathcal{H}) \cap \mathrm{mod}_{sp} K\mathcal{H}$ (see the proof of Lemma 3.6 in [30]).

The above remains also valid if the poset $\mathcal{H}$ is peak-reducible to a hypercritical poset listed in [30, Table 1] (apply the arguments used in the proof of Theorem 3.16 in [31]).

Consider the pair of fully faithful K-linear functors

$$\mathrm{mod} \begin{pmatrix} K & K^3 \\ 0 & K \end{pmatrix} \xrightarrow{T_{V,\widehat{U}}} \mathrm{mod}_{sp} K\mathcal{H} \xrightarrow{T_{\mathcal{H}}} \mathrm{mod}_{sp} KI,$$

where $T_{\mathcal{H}}$ is the subposet induced functor [31, (3.13)]. Since there is a fully faithful K-linear exact functor $\mathrm{modf}\, k\langle t_1, t_2 \rangle \longrightarrow \mathrm{mod} \begin{pmatrix} K & K^3 \\ 0 & K \end{pmatrix}$, see [5] and [50, Section 14.2], then the category $\mathrm{mod}_{sp} KI$ is weak fully K-wild and, according to Lemma 2.7, it is K-endo-wild. □

Remark 3.5. We do not know if the K-endo-wildness of $\mathrm{mod}_{sp}(KI)$ implies that the categories $\mathrm{mod}_{sp}(KI)$ and $\mathrm{prin}(KI)$ are fully K-wild.

4. The pure semisimplicity problem

It is well known that a ring R is of finite representation type if and only if R is right pure semisimple and R is left pure semisimple (see [2], [40], [44], [46], [64]). However, the following **pure semisimplicity conjecture**

($\mathbf{pss}_R$) *A right pure semisimple ring R is of finite representation type*

remains an open problem (see [3] and [46], [49], [53], [54]). The problem is related with the following old problem of Köthe [33]:

"Give a characterisation of unitary rings R such that every left R-module and every right R-module is a direct sum of cyclic modules", see Ringel [37] for more details.

In [27], Ivo Herzog proves the conjecture ($\mathbf{pss}_R$) for arbitrary quasi-Frobenius ring R and for arbitrary PI-ring R. An alternative proof of ($\mathbf{pss}_R$), for any PI-ring R, was given later by Krause [34] and by Schmidmeier [42].

It was shown by the author in Corollaries 3.16 and 5.1 of [53] that the conjecture ($\mathbf{pss}_R$) has a positive solution for all rings R if and only if for any pair of division rings F, G and any simple F-G-bimodule ${}_FM_G$ such that $\dim M_G$ is finite and $\dim_F M = \infty$ one can construct an indecomposable right module of infinite length over the hereditary right artinian ring

$$R_M = \begin{pmatrix} F & {}_FM_G \\ 0 & G \end{pmatrix}$$

or equivalently, one can construct a sequence $X_1 \xrightarrow{f_1} X_2 \to \cdots \to X_m \xrightarrow{f_m} X_{m+1} \to \cdots$, where $X_1, X_2, \ldots$ are indecomposable right R_M-modules of finite length and $f_1, f_2, \ldots$ are non-isomorphisms such that

$$f_m f_{m-1} \cdots f_2 f_1 \neq 0$$

for any $m > 1$.

It follows that if there is a counter-example R to the pure semisimplicity conjecture, then there exists a hereditary counter-example of the form R_M, where F, G are division rings and ${}_FM_G$ is a simple F-G-bimodule such that $\dim M_G$ is finite, $\dim_F M$ is infinite and the infinite dimension-sequence $\mathbf{d}_{-\infty}({}_FG_G)$ associated to the bimodule ${}_FG_G$ in [58, 4.2] satisfies the conditions [54, Proposition 3.1 (b)].

On the other hand it was shown in [53] how a construction of a counter-example R to the pure semisimplicity conjecture depends on a generalized Artin problem for division ring extensions, which is much more difficult than Artin′s problem for division ring extensions solved by Cohn in [7] and by Schofield in [43] (see also [55, Section 2], [57]).

It was shown in [53] how a problem of finding a counter-example to the pure semisimplicity conjecture $(\mathbf{pss}_R)$ formulated below depends on the existence of a division ring embedding $F \subseteq G$ such that $\dim_F G$ is infinite, $\dim G_F$ is finite and countably many additional dimension conditions are satisfied. In other words the existence of such an embedding $F \subseteq G$ is a generalized version of an Artin problem for division ring extensions solved in [7], [43] (see also [55, Section 2] and [57]).

In particular, it was shown in [54] that the hereditary ring

$$R_G = \begin{pmatrix} F & G \\ 0 & G \end{pmatrix}$$

is a counter-example to the pure semisimplicity conjecture, if $F \subseteq G$ is a pair of division rings such that the infinite dimension-sequence $\mathbf{d}_{-\infty}({}_F G_G)$ of the bimodule ${}_F G_G$ (see [58, (4.2)]) is the sequence

$$\boldsymbol{\omega} = (\ldots, 2, 2, \ldots, 2, 2, 1, \infty)$$

In the recent paper [58] we develop a technique introduced in [54] for constructing a class of potential counter-examples to the pure semisimplicity conjecture by means of suitable division ring extensions. We construct there a set $\mathcal{DS}_{pss}$ of cardinality $2^{\aleph_0}$ consisting of sequences

$$v = (\ldots, v_{-m}, v_{-m+1}, \ldots, v_{-2}, v_{-1}, v_0, \infty)$$

with $v_j \in \mathbb{N}$, (see [58, Definition 4.4]) in such a way that the hereditary right artinian ring R_M shown in (2.3) is a counter-example to the pure semisimplicity conjecture, if F and G is a pair of division rings, and ${}_F M_G$ is a F-G-bimodule such that the infinite dimension-sequence $\mathbf{d}_{-\infty}({}_F M_G)$ of the bimodule ${}_F M_G$ (see [58, (4.2)]) belongs to $\mathcal{DS}_{pss}$. Since the set $\mathcal{DS}_{pss}$ is of cardinality $\mathfrak{c} = 2^{\aleph_0}$ the result can observably help in finding a suitable pair of division rings F, G and an F-G-bimodule ${}_F M_G$ such that $\mathbf{d}_{-\infty}({}_F M_G) \in \mathcal{DS}_{pss}$ and the ring R_M is a counter-example to the pure semisimplicity conjecture. On the other hand, this result might discourage people from seeing an easy solution of the pure semisimplicity conjecture.

The reader is referred to [55, Section 2] and [58] for a brief introduction to Artin′s problems and related problems in the representation theory of artinian rings. We use here the notation introduced in [58] (see also [4], [50]).

The following problem was stated in [58].

Problem 4.1. *Assume that F, G are division rings, ${}_FM_G$ is F-G-bimodule such that the associated infinite dimension-sequence $\mathbf{d}_{-\infty}({}_FM_G)$ belongs to $\mathcal{DS}_{pss} = \mathcal{DS}^{(1)}_{pss} \cup \mathcal{DS}^{(2)}_{pss}$. In relation to questions of Auslander* [3, p.11] *the following two problems arise.*

(a) *Find a decomposition of the right R_M-module*

$$L(\mathcal{Q}_M) = \prod_{m=0}^{\infty} Q^{(0)}_m / \bigoplus_{m=0}^{\infty} Q^{(0)}_m \tag{4.2}$$

in a direct sum of indecomposable modules, where $Q^{(0)}_1, Q^{(0)}_2, \ldots, Q^{(0)}_m, \ldots$ is the complete set, up to isomorphism, of the preinjective R_M-modules shown in [58, (2.8)].

(b) *Give a characterization of F-G-bimodules ${}_FM_G$ for which the R_M-module $L(\mathcal{Q}_M)$ is projective.* □

We note that if ${}_FM_G$ is an F-G-bimodule as in 4.1 and satisfies the condition $\dim_F M = \infty$, then the hereditary ring R_M is right pure semisimple and representation-infinite. Hence the R_M-module $L(\mathcal{Q}_M)$ is a direct sum of indecomposable modules and, according to [3, Proposition 2.4], $L(\mathcal{Q}_M)$ has an indecomposable projective direct summand, because, in view of the condition $\dim_F M = \infty$, every indecomposable module in mod (R_M) is either preinjective or projective (see [58, Proposition 4.17]).

A partial answer to the Problem 4.1 (b) was recently given by Okoh [35]. It is shown there that the R_M-module $L(\mathcal{Q}_M)$ is projective, for every F-G-bimodule ${}_FM_G$ such that $\dim_F M = \infty$ and the dimension-sequence $\mathbf{d}_{-\infty}({}_FM_G)$ belongs to $\mathcal{DS}_{pss} = \mathcal{DS}^{(1)}_{pss} \cup \mathcal{DS}^{(2)}_{pss}$. It follows that the module $L(\mathcal{Q}_M)$ is a direct sum of copies of two indecomposable projective right R_M-modules: the simple direct summand $P_0 = \begin{pmatrix} 0 & 0 \\ 0 & G \end{pmatrix}$ of R_M and the non-simple one $P_1 = \begin{pmatrix} F & {}_FM_G \\ 0 & 0 \end{pmatrix}$. Unfortunately, the multiplicities the modules P_0 and P_1 appear in $L(\mathcal{Q}_M)$, as direct summands, are not yet determined.

It would be also interesting to know if the condition $\dim_F M = \infty$ is necessary for the projectivity of the module $L(\mathcal{Q}_M)$, in case the dimension-sequence $\mathbf{d}_{-\infty}({}_FM_G)$ belongs to $\mathcal{DS}_{pss} = \mathcal{DS}^{(1)}_{pss} \cup \mathcal{DS}^{(2)}_{pss}$.

There is a close connection of the pure semisimplicity conjecture to the following product conjecture of Okoh [35] (see also [3]).

Product Conjecture 4.3. *Assume that R is an arbitrary ring with an identity element and let $\{M_j\}_{j\in\mathcal{Z}}$ be an infinite family of pairwise non-isomorphic indecomposable finitely generated right R-modules. Then the* *product* $\prod_{j\in\mathcal{Z}} M_j$ *is not a direct sum of indecomposable finitely generated right* R*-modules.*

Note that the product conjecture for an artinian ring R implies the pure semisimplicity conjecture for R.

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